# A formula for the Θ-invariant from Heegaard diagrams

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September 17, 2012

**Abstract**

The Θ-invariant is the simplest 3-manifold invariant defined with configuration space integrals. It is actually an invariant of rational homology spheres equipped with a combing over the complement of a point. It can be computed as the algebraic intersection of three propagators associated to a given combing $X$ in the 2-point configuration space of a $\mathbb{Q}$–sphere $M$. These propagators represent the linking form of $M$ so that $\Theta(M,X)$ can be thought of as the cube of the linking form of $M$ with respect to the combing $X$. The invariant Θ is the sum of $6\lambda(M)$ and $\frac{p_1(X)}{4}$, where $\lambda$ denotes the Casson-Walker invariant, and $p_1$ is an invariant of combings that is an extension of a first relative Pontrjagin class. In this article, we present explicit propagators associated with Heegaard diagrams of a manifold, and we use these "Morse propagators", constructed with Greg Kuperberg, to prove a combinatorial formula for the Θ-invariant in terms of Heegaard diagrams.

**Keywords:** configuration space integrals, finite type invariants of 3-manifolds, homology spheres, Heegaard splittings, Heegaard diagrams, combings, Casson-Walker invariant, perturbative expansion of Chern-Simons theory, Θ-invariant
**MSC:** 57M27 57N10 55R80 57R20

# 1 Introduction

In this article, a $\mathbb{Q}$–*sphere* or *rational homology sphere* is a smooth closed oriented 3-manifold that has the same rational homology as $S^3$.

## 1.1 General introduction

The work of Witten [Wit89] pioneered the introduction of many $\mathbb{Q}$–sphere invariants, among which the Le-Murakami-Ohtsuki universal finite type invariant [LMO98] and the Kontsevich

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configuration space invariant [Kon94] that was proved to be equivalent to the LMO invariant for integral homology spheres by G. Kuperberg and D. Thurston [KT99]. The construction of the Kontsevich configuration space invariant for a $\mathbb{Q}$–sphere $M$ involves a point $\infty$ in $M$, an identification of a neighborhood of $\infty$ with a neighborhood of $\infty$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and a parallelization $\tau$ of $(\check{M} = M \setminus \{\infty\})$ that coincides with the standard parallelization of $\mathbb{R}^3$ near $\infty$. The Kontsevich configuration space invariant is in fact an invariant of $(M, \tau)$. Its degree one part $\Theta(M, \tau)$ is the sum of $6\lambda(M)$ and $\frac{p_1(\tau)}{4}$, where $\lambda$ is the Casson-Walker invariant and $p_1$ is a Pontrjagin number associated with $\tau$, according to a Kuperberg Thurston theorem [KT99] generalized to rational homology spheres in [Les04b]. Here, the Casson-Walker invariant $\lambda$ is normalized like in [AM90, GM92, Mar88] for integral homology spheres, and like $\frac{1}{2}\lambda_W$ for rational homology spheres where $\lambda_W$ is the Walker normalisation in [Wal92].

The invariant $\Theta(M, \tau)$ reads

$$\Theta(M, \tau) = \int_{\check{M}^2 \setminus \mathrm{diag}(\check{M})^2} \omega(M, \tau)^3$$

for some closed 2-form $\omega(M, \tau)$ that is often called a *propagator*. As it is developed in [Les04b, Section 6.5], $\Theta(M, \tau)$ can also be written as the algebraic intersection of three 4-dimensional chains in a compactification $C_2(M)$ of $\check{M}^2 \setminus \mathrm{diag}(\check{M})^2$, for chains that are Poincaré dual to $\omega(M, \tau)$ in the 6–dimensional configuration space $C_2(M)$. In this article, a *propagator* will be such a 4-chain. For more precise definitions, see Subsection 1.4. A *combing* of a 3-manifold $M$ as above is an asymptotically constant nonzero section of the tangent bundle of $\check{M}$.

In Theorem 1.1, we will prove that the invariant $\Theta$ is an invariant of combed $\mathbb{Q}$–spheres $(M, X)$ rather than an invariant of parallelised punctured $\mathbb{Q}$–spheres, so that $(4\Theta(M, X) - 24\lambda(M))$ is an extension of the Pontrjagin number $p_1$ to combings. This invariant $p_1$ of combings is studied in [Les12b], and it is shown to be the analogue of the Gompf $\theta$-invariant [Gom98, Section 4] of $\mathbb{Q}$–sphere combings, for asymptotically constant combings of punctured $\mathbb{Q}$–spheres. The variations of $\Theta$, $\theta$ and $p_1$ under various combing changes are described in [Les12b].

In Section 2, we describe explicit propagators associated with Morse functions or with Heegaard splittings. These "Morse propagators" have been obtained in collaboration with Greg Kuperberg. Then we use these propagators to produce a combinatorial description of $\Theta$ in terms of Heegaard splittings in Theorem 1.5.

Our Morse propagators and our techniques could be applied to compute more configuration space invariants, and they might be useful to relate finite type invariants to Heegaard Floer homology.

This article benefited from the stimulating visit of Greg Kuperberg in Grenoble in 2010-2011.

## 1.2 Conventions and notations

Unless otherwise mentioned, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients chains or

manifolds. The fiber of the normal bundle $N(A)$ of an oriented submanifold $A$ is oriented so that the normal bundle followed by the tangent bundle of the submanifold induce the orientation of the ambient manifold, fiberwise. The transverse intersection of two submanifolds $A$ and $B$ is oriented so that the normal bundle of $A\cap B$ is $(N(A)\oplus N(B))$, fiberwise.

## 1.3 On configuration spaces

Here, *blowing up* a submanifold $A$ means replacing it by its unit normal bundle. Locally, $((\mathbb{R}^c=\{0\}\cup]0,\infty[\times S^{c-1})\times A)$ is replaced by $([0,\infty[\times S^{c-1}\times A)$. Topologically, this amounts to removing an open tubular neighborhood of the submanifold (thought of as infinitely small), but the process is canonical, so that the created boundary is the unit normal bundle of the submanifold and there is a canonical projection from the manifold obtained by blow-up to the initial manifold.

In a closed 3-manifold $M$, we shall fix a point $\infty$ and define the blown-up manifold $C_1(M)$ as the compact 3-manifold obtained from $M$ by blowing up $\{\infty\}$. This space $C_1(M)$ is a compactification of $\check{M}=(M\setminus\{\infty\})$.

The *configuration space* $C_2(M)$ is the compact 6–manifold with boundary and corners obtained from $M^2$ by blowing up $(\infty,\infty)$, and the closures of $\{\infty\}\times\check{M}$, $\check{M}\times\{\infty\}$ and the diagonal of $\check{M}^2$, successively.

Then the boundary $\partial C_2(M)$ of $C_2(M)$ contains the unit normal bundle of the diagonal of $\check{M}^2$. This bundle is canonically isomorphic to the unit tangent bundle $U\check{M}$ via the map

$$[(x,y)]\in\frac{\frac{T_m\check{M}^2}{\mathrm{diag}}\setminus\{0\}}{\mathbb{R}^{+*}}\mapsto[y-x]\in\frac{T_m\check{M}\setminus\{0\}}{\mathbb{R}^{+*}}.$$

When $M$ is a rational homology sphere, the configuration space $C_2(M)$ has the same rational homology as $S^2$ and $H_2(C_2(M);\mathbb{Q})$ has a canonical generator $[S]$ that is the homology class of a product $(x\times\partial B(x))$ where $B(x)$ is a ball embedded in $\check{M}$ that contains $x$ in its interior. For a 2-component link $(J,K)$ of $M$, the homology class $[J\times K]$ of $J\times K$ in $H_2(C_2(M);\mathbb{Q})$ reads $lk(J,K)[S]$, where $lk(J,K)$ is the linking number of $J$ and $K$.

## 1.4 On propagators

When $M$ is a rational homology sphere, a *propagator* of $C_2(M)$ is a 4–cycle $F$ of $(C_2(M),\partial C_2(M))$ that is Poincaré dual to the preferred generator of $H^2(C_2(M);\mathbb{Q})$ that maps $[S]$ to 1. For such a propagator $F$, for any 2-cycle $G$ of $C_2(M)$,

$$[G]=\langle F,G\rangle_{C_2(M)}[S]$$

in $H_2(C_2(M);\mathbb{Q})$ where $\langle F,G\rangle_{C_2(M)}$ denotes the algebraic intersection of $F$ and $G$ in $C_2(M)$.

Let $B$ and $\frac{1}{2}B$ be two balls in $\mathbb{R}^3$ of respective radii $R$ and $\frac{R}{2}$, centered at the origin in $\mathbb{R}^3$. Identify a neighborhood of $\infty$ in $M$ with $S^3\setminus(\frac{1}{2}B)$ in $(S^3=\mathbb{R}^3\cup\{\infty\})$ so that $\check{M}$ reads

$\check{M} = B_M \cup_{]R/2,R]\times S^2} (\mathbb{R}^3 \setminus (\frac{1}{2}B))$ for a rational homology ball $B_M$ whose complement in $\check{M}$ is identified with $\mathbb{R}^3 \setminus B$. There is a canonical regular map

$$p_\infty\colon (\partial C_2(M) \setminus UB_M) \to S^2$$

that maps the limit in $\partial C_2(M)$ of a sequence of ordered pairs of distinct points of $\left(\check{M} \setminus B_M\right)^2$ to the limit of the direction from the first point to the second one. See [Les04a, Lemma 1.1]. Let

$$\tau_s\colon \mathbb{R}^3 \times \mathbb{R}^3 \to T\mathbb{R}^3$$

denote the standard parallelization of $\mathbb{R}^3$. In this article, a *combing* $X$ of a $\mathbb{Q}$–sphere $M$ is a section of $U\check{M}$ that is constant outside $B_M$, i.e. that reads $\tau_s((\check{M} \setminus B_M) \times \{V(X)\})$ for some fixed $V(X) \in S^2$ outside $B_M$. Then the *propagator boundary* $\partial F_X$ associated with such a combing $X$ is the following 3–cycle of $\partial C_2(M)$

$$\partial F_X = p_\infty^{-1}(V(X)) \cup X(B_M)$$

where the restriction of the combing $X$ to $B_M$ is the part $X(B_M)$ of $\partial C_2(M)$ and a *propagator associated with the combing* $X$ is a 4–chain $F_X$ of $C_2(M)$ whose boundary reads $\partial F_X$. Such an $F_X$ is indeed a propagator (because for a tiny sphere $\partial B(x)$ around a point $x$, $\langle x \times \partial B(x), F_X\rangle_{C_2(M)}$ is the algebraic intersection in $U\check{M}$ of a fiber and the section $X(\check{M})$, that is one).

## 1.5 On the $\Theta$-invariant of a combed $\mathbb{Q}$–sphere

**Theorem 1.1** *Let $X$ be a combing of a rational homology sphere $M$, and let $(-X)$ be the opposite combing. Let $F_X$ and $F_{-X}$ be two associated transverse propagators. Then $F_X \cap F_{-X}$ is a two-dimensional cycle whose homology class is independent of the chosen propagators. It reads $\Theta(M,X)[S]$, where $\Theta(M,X)$ is therefore a rational valued topological invariant of $M$ and of the homotopy class of $X$.*

Proof: Let us first show that $C_2(M)$ has the same rational homology as $S^2$. The space $C_2(M)$ is homotopy equivalent to $(\check{M}^2 \setminus \mathrm{diag})$. Since $\check{M}$ is a rational homology $\mathbb{R}^3$, the rational homology of $(\check{M}^2 \setminus \mathrm{diag})$ is isomorphic to the rational homology of $((\mathbb{R}^3)^2 \setminus \mathrm{diag})$. Since $((\mathbb{R}^3)^2 \setminus \mathrm{diag})$ is homeomorphic to $\mathbb{R}^3 \times ]0,\infty[ \times S^2$ via the map

$$(x,y) \mapsto (x, \| y-x \|, \frac{1}{\| y-x \|}(y-x)),$$

$((\mathbb{R}^3)^2 \setminus \mathrm{diag})$ is homotopy equivalent to $S^2$.

In particular, since $H_3(C_2(M);\mathbb{Q}) = 0$, there exist (transverse) propagators $F_X$ and $F_{-X}$ with the given boudaries $\partial F_X$ and $\partial F_{-X}$. Without loss, assume that $F_{\pm X} \cap \partial C_2(M) = \partial F_{\pm X}$. Since $\partial F_X$ and $\partial F_{-X}$ do not intersect, $F_X \cap F_{-X}$ is a 2–cycle. Since $H_4(C_2(M);\mathbb{Q}) = 0$, the homology class of $F_X \cap F_{-X}$ in $H_2(C_2(M);\mathbb{Q})$ does not depend on the choices of $F_X$ and $F_{-X}$

with their given boundaries. Then it is easy to see that $\Theta(M,X) \in \mathbb{Q}$ is a locally constant function of the combing $X$. $\diamond$

When $M$ is an integral homology sphere, a combing $X$ is the first vector of a unique parallelization $\tau(X)$ that coincides with $\tau_s$ outside $B_M$, up to homotopy. When $M$ is a rational homology sphere, and when $X$ is the first vector of a such a parallelization $\tau(X)$, this parallelization is again unique. In this case, the invariant $\Theta(M,X)$ is the degree 1 part of the Kontsevich invariant of $(M,\tau(X))$ [Kon94, KT99, Les04a]. Let $W$ be a connected compact 4–dimensional manifold with corners with signature 0 whose boundary

$$\partial W = B_M \cup_{1\times\partial B_M} (-[0,1]\times S^2) \cup_{0\times S^2} (-B^3).$$

is identified with an open subspace of one of the products $[0,1[\times B^3$ or $]0,1]\times B_M$ near $\partial W$. Then the Pontrjagin number $p_1(\tau(X))$ is the obstruction to extending the trivialization of $TW\otimes\mathbb{C}$ induced by $\tau(X)$ and $\tau_s$ on $\partial W$ to $W$. This obstruction lives in $H^4(W,\partial W;\pi_3(SU(4))=\mathbb{Z})=\mathbb{Z}$. See [Les04a, Section 1.5] for more details. In [KT99], G. Kuperberg and D. Thurston proved that

$$\Theta(M,X) = 6\lambda(M) + \frac{p_1(\tau(X))}{4}$$

when $M$ is an integral homology sphere. This result was extended to $\mathbb{Q}$–spheres by the author in [Les04b, Theorem 2.6 and Section 6.5]. Setting $p_1(X) = (4\Theta(M,X) - 24\lambda(M))$ extends the Pontrjagin number from parallelizations to combings so that the formula above is still valid for combings.

The following theorem is proved in [Les12b].

**Theorem 1.2** *Let $X$ and $Y$ be two combings of $M$ such that the cycle $\partial F_Y$ is transverse to $\partial F_X$ and to $\partial F_{-X}$ in $\partial C_2(M)$. Then the oriented intersection $\partial F_X \cap \partial F_Y$ (resp. $\partial F_X \cap \partial F_{-Y}$) is a section of $U\check{M}$ over an oriented link $L_{X=Y}$ (resp. $L_{X=-Y}$) and*

$$\Theta(M,Y) - \Theta(M,X) = \frac{p_1(Y)-p_1(X)}{4} = lk(L_{X=Y}, L_{X=-Y}).$$

## 1.6 On Heegaard diagrams

Every closed 3–manifold $M$ can be written as the union of two handlebodies $H_A$ and $H_B$ glued along their common boundary that is a genus $g$ surface as

$$M = H_A \cup_{\partial H_A} H_B$$

where $\partial H_A = -\partial H_B$. Such a decomposition is called a *Heegaard decomposition* of $M$. A *system of meridian disks* for $H_A$ is a system of $g$ disjoint disks $D(\alpha_i)$ properly embedded in $H_A$ such that the union of the boundaries $\alpha_i$ of the $D(\alpha_i)$ does not separate $\partial H_A$. Let $(D(\alpha_i))_{i\in\{1,\dots,g\}}$ be such a system for $H_A$ and let $(D(\beta_j))_{j\in\{1,\dots,g\}}$ be such a system for $H_B$. Then the surface equipped with the collections of the curves $\alpha_i$ and the curves $\beta_j = \partial D(B_j)$ determines $M$. When the

collections $(\alpha_i)_{i\in\{1,\dots,g\}}$ and $(\beta_j)_{j\in\{1,\dots,g\}}$ are transverse, the data $(\partial H_A, (\alpha_i)_{i\in\{1,\dots,g\}}, (\beta_j)_{j\in\{1,\dots,g\}})$ is called a *Heegaard diagram.*

We fix such a diagram. A *crossing* $c$ of the diagram is an intersection point of a curve $\alpha_{i(c)}$ and a curve $\beta_{j(c)}$. Its sign $\sigma(c)$ is 1 if $\partial H_A$ is oriented by the oriented tangent vector of $\alpha_i$ followed by the oriented tangent vector of $\beta_j$ at $c$. It is $(-1)$ otherwise. The collection of crossings is denoted by $\mathcal{C}$.

Fix a point $a_i$ inside each disk $D(\alpha_i)$ and a point $b_j$ inside each disk $D(\beta_j)$. Then join $a_i$ to each crossing $c$ of $\alpha_i$ by a segment $[a_i, c]_{D(\alpha_i)}$ oriented from $a_i$ to $c$ in $D(\alpha_i)$, so that these segments only meet at $a_i$ for different $c$. Similarly define segments $[c, b_{j(c)}]_{D(\beta_{j(c)})}$ from $c$ to $b_{j(c)}$ in $D(\beta_{j(c)})$. Then for each $c$, define the *flow line* $\gamma(c) = [a_{i(c)}, c]_{D(\alpha_{i(c)})} \cup [c, b_{j(c)}]_{D(\beta_{j(c)})}$.

For good choices of the above segments, this flow line is the closure of an actual flow line associated with a Morse function giving birth to this diagram that will be discussed in Section 2.

### 1.7 Parallels of flow lines

For each point $a_i$, choose a point $a_i^+$ and a point $a_i^-$ close to $a_i$ outside $D(\alpha_i)$ so that $a_i^+$ is on the positive side of $D(\alpha_i)$ (the side of the positive normal) and $a_i^-$ is on the negative side of $D(\alpha_i)$. Similarly fix points $b_j^+$ and $b_j^-$ close to the $b_j$ and outside the $D(\beta_j)$.

Then for a crossing $c \in \alpha_{i(c)} \cap \beta_{j(c)}$, $\gamma(c)_\parallel$ will denote the following chain. Consider a small meridian curve $m(c)$ of $\gamma(c)$ on $\partial H_A$, it intersects $\beta_{j(c)}$ at two points: $c_A^+$ on the positive side of $D(\alpha_{i(c)})$ and $c_A^-$ on the negative side of $D(\alpha_{i(c)})$. The meridian $m(c)$ also intersects $\alpha_{i(c)}$ at $c_B^+$ on the positive side of $D(\beta_{j(c)})$ and $c_B^-$ on the negative side of $D(\beta_{j(c)})$. Let $[c_A^+, c_B^+]$, $[c_A^+, c_B^-]$, $[c_A^-, c_B^+]$ and $[c_A^-, c_B^-]$ denote the four quarters of $m(c)$ with the natural ends and orientations associated with the notation.


Figure 1: $m(c)$, $c_A^+$, $c_A^-$, $c_B^+$ and $c_B^-$

Let $\gamma_A^+(c)$ (resp. $\gamma_A^-(c)$) be an arc parallel to $[a_{i(c)}, c]_{D(\alpha_{i(c)})}$ from $a_{i(c)}^+$ to $c_A^+$ (resp. from $a_{i(c)}^-$ to $c_A^-$) that does not meet $D(\alpha_{i(c)})$. Let $\gamma_B^+(c)$ (resp. $\gamma_B^-(c)$) be an arc parallel to $[c, b_{j(c)}]_{D(\beta_{j(c)})}$ from $c_B^+$ to $b_{j(c)}^+$ (resp. from $c_B^-$ to $b_{j(c)}^-$) that does not meet $D(\beta_{j(c)})$.

$$\gamma(c)_\parallel = \frac{1}{2}(\gamma_A^+(c) + \gamma_A^-(c)) + \frac{1}{4}([c_A^+, c_B^+] + [c_A^+, c_B^-] + [c_A^-, c_B^+] + [c_A^-, c_B^-]) + \frac{1}{2}(\gamma_B^+(c) + \gamma_B^-(c)).$$

Set $a_{i\parallel} = \frac{1}{2}(a_i^+ + a_i^-)$ and $b_{j\parallel} = \frac{1}{2}(b_j^+ + b_j^-)$. Then $\partial\gamma(c)_\parallel = b_{j(c)\parallel} - a_{i(c)\parallel}$.

## 1.8 A 2-cycle $C_h$ of $C_2(M)$ associated with a Heegaard diagram

Let

$$[\mathcal{J}_{ji}]_{(j,i)\in\{1,\dots,g\}^2} = [\langle \alpha_i, \beta_j\rangle_{\partial H_A}]^{-1}$$

be the inverse matrix of the intersection matrix.

**Proposition 1.3** *Set*

$$C_h = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\gamma(c)\times\gamma(d)_{\|}) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(\gamma(c)\times\gamma(c)_{\|}).$$

*Then $C_h$ is a 2–cycle of $C_2(M)$. Its homology class $[C_h]$ depends neither on the orientations of the $\alpha_i$ and the $\beta_j$, nor on their order. Permuting the roles of the $\alpha_i$ and the roles of the $\beta_j$ does not change it either.*

Proof: Let us first prove that $C_h$ is a 2-cycle. Note that, for any $j$,

$$\sum_{c\in\beta_j} \mathcal{J}_{j(d)i(c)}\sigma(c) = \sum_{i=1}^{g} \mathcal{J}_{j(d)i}\langle\alpha_i,\beta_j\rangle = \delta_{jj(d)}$$

and, for any $i$, $\sum_{c\in\alpha_i} \mathcal{J}_{j(c)i(d)}\sigma(c) = \sum_{j=1}^{g} \mathcal{J}_{ji(d)}\langle\alpha_i,\beta_j\rangle = \delta_{ii(d)}$. Therefore, for any $d\in\mathcal{C}$,

$$\partial\left(\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\gamma(c)\right) = \mathcal{J}_{j(d)i(d)}(b_{j(d)} - a_{i(d)}) = \mathcal{J}_{j(d)i(d)}\partial\gamma(d)$$

and

$$\begin{aligned}\partial C_h = \; & \textstyle\sum_{d\in\mathcal{C}} \sigma(d)\mathcal{J}_{j(d)i(d)}(\partial\gamma(d))\times\gamma(d)_{\|} - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(\partial\gamma(c))\times\gamma(c)_{\|} \\ & \textstyle - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)\times\partial\gamma(c)_{\|} + \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)\times\partial\gamma(c)_{\|} = 0.\end{aligned}$$

In particular our choices for the $a_i^{\pm}$ near the $a_i$ (resp. for the $b_j^{\pm}$) do not matter as soon as they satisfy our assumptions of being on the wanted side of $D(\alpha_i)$ (resp. $D(\beta_j)$). Now, since the $+$ and the $-$ play the same roles in the formula, $\gamma(c)_{\|}$ does not depend on the orientations of the $\alpha_i$ and the $\beta_j$. Since changing the orientation of $\alpha_{i(c)}$ leaves $\mathcal{J}_{j(d)i(c)}\sigma(c)$ invariant and changing the orientation of $\beta_{j(c)}$ leaves $\mathcal{J}_{j(c)i(d)}\sigma(c)$ invariant, the cycle $C_h$ does not depend on the orientations of the $\alpha_i$ and the $\beta_j$. It clearly does not depend on the numbering. It is also easy to see that permuting the roles of the $\alpha_i$ and the $\beta_j$ reverses the orientations of the $\gamma(c)$, changes $\mathcal{J}$ to the transposed matrix and does not change the cycle $C_h$ either. $\diamond$

Define the rational number $\lambda_h$ associated with our Heegaard diagram by

$$[C_h] = \lambda_h[S].$$

Note that $\lambda_h$ is additive under connected sum of Heegaard diagrams, and therefore it is invariant under stabilisation of diagrams, but, as it is easily shown in [Les12a], it is not an invariant of Heegaard splittings. In the next subsection, we state Proposition 1.4 that yields a combinatorial formula for $\lambda_h$.

## 1.9 Evaluating some 2–cycles of $C_2(M)$.

When $d$ and $e$ are two crossings of $\alpha_i$, $[d,e]_{\alpha_i} = [d,e]_\alpha$ denotes the set of crossings from $d$ to $e$ (including them) along $\alpha_i$, or the closed arc from $d$ to $e$ in $\alpha_i$ depending on the context. Then $[d,e[_\alpha = [d,e]_\alpha \setminus \{e\}$, $]d,e]_\alpha = [d,e]_\alpha \setminus \{d\}$ and $]d,e[_\alpha = [d,e[_\alpha \setminus \{d\}$.

Now, for such a part $I$ of $\alpha_i$,

$$\langle I, \beta_j \rangle = \sum_{c \in I \cap \beta_j} \sigma(c).$$

We shall also use the notation $|$ for ends of arcs to say that an end is half-contained in an arc, and that it must be counted with coefficient $1/2$. ("$[d,e|_\alpha = [d,e]_\alpha \setminus \{e\}/2$"). We agree that $|d,d|_\alpha = \emptyset$.

We use the same notation for arcs $[d,e|_{\beta_j} = [d,e|_\beta$ of $\beta_j$. For example, if $d$ is a crossing of $\alpha_i \cap \beta_j$, then

$$\langle [d,d|_\alpha, \beta_j \rangle = \frac{\sigma(d)}{2}$$

and

$$\langle [c,d|_\alpha, [e,d|_\beta \rangle = \frac{\sigma(d)}{4} + \sum_{c \in [c,d[_\alpha \cap [e,d[_\beta} \sigma(c).$$

The following proposition is proved in Subsection 2.3.

**Proposition 1.4** *For any curve $\alpha_i$ (resp. $\beta_j$), choose a basepoint $c(\alpha_i)$ (resp. $c(\beta_j)$). These choices being made, for two crossings $c$ and $d$ of $\mathcal{C}$, set*

$$\ell(c,d) = \langle [c(\alpha(c)), c|_\alpha, [c(\beta(d)), d|_\beta \rangle - \sum_{(i,j) \in \{1,\dots,g\}^2} \mathcal{J}_{ji} \langle [c(\alpha(c)), c|_\alpha, \beta_j \rangle \langle \alpha_i, [c(\beta(d)), d|_\beta \rangle$$

*where $\alpha(c) = \alpha_{i(c)}$ and $\beta(c) = \beta_{j(c)}$. Then, for any 2–cycle $D = \sum_{(c,d) \in \mathcal{C}^2} f_{cd}(\gamma(c) \times \gamma(d)_{\parallel})$ of $C_2(M)$,*

$$[D] = \sum_{(c,d) \in \mathcal{C}^2} f_{cd} \ell(c,d)[S] = \sum_{(c,d) \in \mathcal{C}^2} f_{cd} \ell(d,c)[S].$$

## 1.10 Combed Heegaard diagrams

Select $g$ crossings $c_i \in \alpha_i \cap \beta_{\rho^{-1}(i)}$, for a permutation $\rho$ of $\{1,2,\dots,g\}$, so that $\gamma_i = \gamma(c_i)$ goes from $a_i$ to $b_{\rho^{-1}(i)}$, and let $\mathcal{P}$ be the set of these selected crossings.

Let

$$L(\mathcal{P}) = \sum_{i=1}^{g} \gamma_i - \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) \gamma(c)$$

and let $L(\mathcal{P})_\parallel = \sum_{i=1}^{g} \gamma_{i\parallel} - \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) \gamma(c)_\parallel$.

Note that $L(\mathcal{P})$ is a cycle since

$$\partial L(\mathcal{P}) = \sum_{i=1}^{g}(b_i - a_i) - \sum_{(i,j)\in\{1,\dots,g\}^2} \mathcal{J}_{ji}\langle \alpha_i, \beta_j\rangle_{\partial H_A}(b_j - a_i) = 0$$

and that $L(\mathcal{P})_{\parallel}$ is also a cycle disjoint from $L$. Also note that Proposition 1.4 yields a combinatorial formula for $lk(L(\mathcal{P}), L(\mathcal{P})_{\parallel})$ since $[L(\mathcal{P}) \times L(\mathcal{P})_{\parallel}] = lk(L(\mathcal{P}), L(\mathcal{P})_{\parallel})[S]$ in $H_2(C_2(M);\mathbb{Q})$. The cycle $L(\mathcal{P})$ depends neither on the orientations of the $\alpha_i$ and the $\beta_j$, nor on their order. Permuting the roles of the $\alpha_i$ and the roles of the $\beta_j$ reverses its orientation and leaves $lk(L(\mathcal{P}), L(\mathcal{P})_{\parallel})$ unchanged.

Select a connected component $W$ of $\partial H_A \setminus (\cup_{i=1}^{g}\alpha_i \cup \cup_{i=1}^{g}\beta_i)$. We shall see in Subsection 3.1 that the choice of $\mathcal{P}$ and $W$ equips $M$ with a combing $X(W,\mathcal{P})$.

The choice of $\mathcal{P}$ being fixed, represent the Heegaard diagrams in a plane by removing a disk of $W$ and by cutting the surface $\partial H_A$ along the $\alpha_i$ so that the crossings different from $c_i$ on $\alpha_i$ are located as far as possible from $c_i$, and so that the arcs of $\beta_j$ are horizontal near their ends, like in Figure 2.

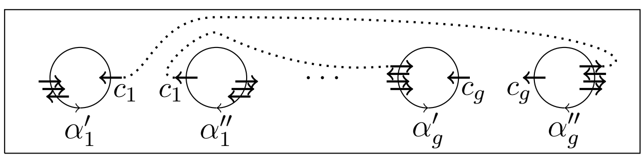


Figure 2: The Heegaard surface cut along the $\alpha_i$

The rectangle has the standard parallelization of the plane. Then there is a map "unit tangent vector" from each partial projection of a beta curve $\beta_j$ in the plane to $S^1$. The total degree of this map for the curve $\beta_j$ is denoted by $d_e(\beta_j)$. For a crossing $c \in \beta_j$, $d_e(|c_{\rho(j)}, c|_\beta) \in \frac{1}{2}\mathbb{Z}$ denotes the degree of the restriction of this map to the arc $|c_{\rho(j)}, c|_\beta$. For any $c \in \mathcal{C}$, define

$$d_e(c) = d_e(|c_{\rho(j(c))}, c|_\beta) - \sum_{(r,s)\in\{1,\dots,g\}^2} \mathcal{J}_{sr}\langle \alpha_r, |c_{\rho(j(c))}, c|_\beta\rangle d_e(\beta_s),$$

where $|c, c|_\beta = \emptyset$. Then set

$$e(W,\mathcal{P}) = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)d_e(c).$$

In Section 5.1, $e(W,\mathcal{P})$ will be identified with an Euler class. See Proposition 5.2.

## 1.11 Statement of the main theorem

The main result of this article is the following theorem.

**Theorem 1.5** *For any Heegaard diagram of a rational homology sphere $M$, for any connected component $W$ of $\partial H_A \setminus (\cup_{i=1}^g \alpha_i \cup \cup_{i=1}^g \beta_i)$, and for any set $\mathcal{P}$ of selected crossings as above*

$$\Theta(M, X(W,\mathcal{P})) = \lambda_h + lk(L(\mathcal{P}), L(\mathcal{P})_\parallel) - e(W,\mathcal{P}).$$

# 2 Propagators associated with Morse functions

In this section, we introduce a propagator associated with a self-indexed Morse function $h$ without minima and maxima of $\check{M}$. This Morse propagator has been constructed in a joint work with Greg Kuperberg.

## 2.1 The Morse function $h$

Start with $\mathbb{R}^3$ equipped with its standard height function $h_0$ and replace $[0,2g] \times [0,4] \times [0,6]$ with a rational homology ball $C_M$ equipped with a Morse function $h$ that coincides with $h_0$ on $\partial\left([0,2g] \times [0,4] \times [0,6]\right)$, and that has $2g$ critical points g points $a_1, \dots, a_g$ of index 1 that are mapped to 1 by $h$, and g points $b_1, \dots, b_g$ of index 2 that are mapped to 5 by $h$. Let $\check{M}$ be the associated open manifold, and let $M$ be its one-point compactification. Equip $\check{M}$ with a Riemannian metric that coincides with the standard one outside $[0,2g] \times [0,4] \times [0,6]$.

The preimage $H_a$ of $]-\infty, 2]$ under $h$ in $C_M$ has the standard representation of the bottom part of Figure 3. Our standard representation of the preimage $H_b$ of $[4, +\infty[$ under $h$ in $C_M$ is shown in the upper part of Figure 3. It can be thought of as the complement of the bottom part in $[0,2g] \times [0,4] \times [0,6]$.

The two-dimensional ascending manifold of $a_i$ is oriented arbitrarily, its closure is denoted by $A_i$. Its intersection with $H_a$ is denoted by $D(\alpha_i)$. The boundary of $D(\alpha_i)$ is denoted by $\alpha_i$. The descending manifold of $a_i$ is made of two half-lines $d(a_i)$ and $e(a_i)$ starting as vertical lines and ending at $a_i$. The one with the orientation of the positive normal to $A_i$ is called $d(a_i)$.

Symmetrically, the two-dimensional descending manifold of $b_j$ is oriented arbitrarily, its closure is denoted by $B_j$. The $B_j$ are assumed to be transverse to the $A_i$ outside the critical points. The ascending manifold of $b_j$ is made of two half-lines $d(b_j)$ and $e(b_j)$ starting at $b_j$ and ending as vertical lines. The one with the orientation of the positive normal to $B_j$ is called $d(b_j)$. See Figure 4.

Let

$$H_{a,2} = C_M \cap h^{-1}(2)$$

and similarly define $H_{b,4} = C_M \cap h^{-1}(4)$. The preimage of $[2,4]$ in $C_M$ is the product $H_{a,2} \times [2,4]$. Its intersection with $A_i$ is $-\alpha_i \times [2,4]$ and its intersection with $B_j$ is $\beta_j \times [2,4]$. Each crossing $c$ of $\alpha_i \cap \beta_j$ has a sign $\sigma(c)$ and an associated flow line $\gamma(c)$ from $a_i$ to $b_j$ oriented as such.

Note the following lemma.

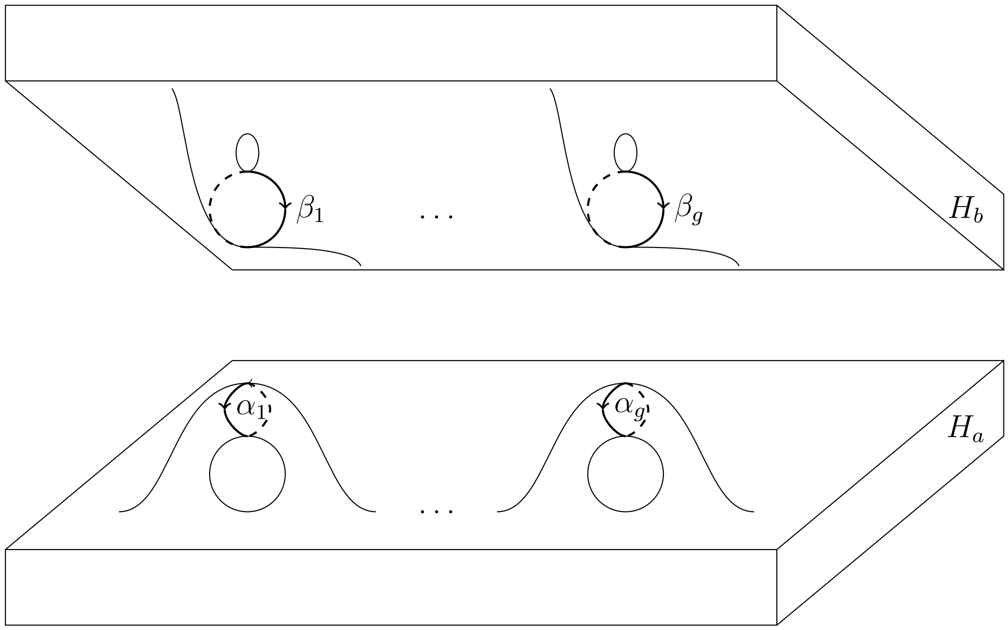

Figure 3: $H_a$ and $H_b$

**Lemma 2.1** *Let $c \in \alpha_i \cap \beta_j$. Along $\gamma(c)$, $A_i$ is cooriented by $\sigma(c)\beta_j$ and $B_j$ is cooriented by $\sigma(c)\alpha_i$.*

$$B_j \cap A_i = \sum_{c \in \alpha_i \cap \beta_j} \sigma(c)\gamma(c).$$

$\diamond$

## 2.2 The propagator associated with $h$

Again, let

$$[\mathcal{J}_{ji}]_{(j,i)\in\{1,\ldots,g\}^2} = [\langle \alpha_i, \beta_j \rangle_{H_a,2}]^{-1}$$

be the inverse matrix of the intersection matrix.

Let $s_\phi(C_M)$ be the closure of the section of $UC_M$ directed by the gradient of $h$ outside the critical points. This closure contains the restriction of the unit tangent bundle to the critical points, up to orientation. Let $\phi$ be the flow associated with the gradient of $h$. Let $F_\phi$ be the closure in $C_2(M)$ of the image of

$$\begin{array}{lcl} \check{M} \setminus \{a_i, b_i; i \in \{1,\ldots,g\}\} \times ]0,+\infty[ & \to & C_2(M) \\ (x,t) & \mapsto & (x, \phi_t(x)) \end{array}$$

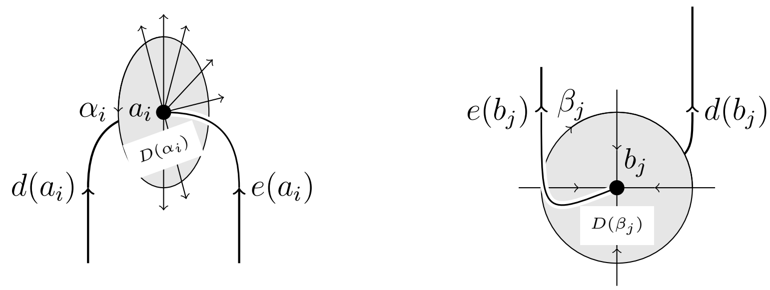


Figure 4: $d(a_i)$, $e(a_i)$, $d(b_j)$, $e(b_j)$

let $((B_j \times A_i) \cap C_2(M))$ denote the closure of $\big((B_j \times A_i) \cap (\check{M}^2 \setminus \text{diagonal})\big)$ in $C_2(M)$ and let

$$F_{\mathcal{I}} = \sum_{(i,j)\in\{1,\dots,g\}^2} \mathcal{J}_{ji} \left((B_j \times A_i) \cap C_2(M)\right).$$

Let $\vec{v}$ be the upward vector in $S^2$, and let

$$\partial_{od} = p_\infty^{-1}(\vec{v}) \cap \left(\partial C_2(M) \setminus U\check{M}\right)$$

be a boundary part *outside the diagonal of* $\check{M}^2$. (If $\vec{v}_\infty$ denotes the upward vertical vector in the boundary of the compactification $C_1(M)$ of $\check{M}$, then $\partial_{od}$ contains $\big(-\check{M} \times \vec{v}_\infty - \big((-\vec{v}_\infty) \times \check{M}\big)\big)$.)

**Theorem 2.2 (Kuperberg–Lescop)** *The 4–chain $(F_\phi + F_{\mathcal{I}})$ is a propagator and its boundary, that lies in $\partial C_2(M)$, is*

$$\partial(F_\phi + F_{\mathcal{I}}) = \partial_{od} + \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) U\check{M}_{|\gamma(c)} + \overline{s_\phi(\check{M})}.$$

PROOF: The expression of $\partial(F_\phi + F_{\mathcal{I}})$ is the immediate consequence of the following two lemmas. Then it is easy to see that for a tiny sphere $\partial B(x)$ around a point $x$ outside the $\gamma(c)$, $\langle (x \times \partial B(x)), F_\phi + F_{\mathcal{I}} \rangle_{C_2(M)}$ will be the algebraic intersection in $U\check{M}$ of a fiber and the section $s_\phi(\check{M})$ that is one. $\diamond$

Note that $U\check{M}_{|\gamma(c)}$ is diffeomorphic to $S^2 \times \gamma(c)$. For simplicity, $U\check{M}_{|\gamma(c)}$ will sometimes be simply denoted by $S^2 \times \gamma(c)$, or by $S^2 \times_\tau \gamma(c)$ when the parallelization $\tau$ that induces such a diffeomorphism matters.

**Lemma 2.3**

$$\partial F_\phi = \partial_{od} + \overline{s_\phi(\check{M})} + \sum_{i=1}^{g} (e(a_i) - d(a_i)) \times A_i + \sum_{j=1}^{g} B_j \times (e(b_j) - d(b_j))$$

PROOF: The boundary of $F_\phi$ is made of $\left(\partial_{od}+\overline{s_\phi(\check{M})}\right)$ and some other parts coming from the critical points. Let us look at the part coming from $a_i$, where the closures $d(a_i)$ and $e(a_i)$ of flow lines stop and closures of flow lines of $A_i$ start. Consider a tubular neighborhood

$$D^2\times d(a_i)=\{(u\exp(i\theta),y);u\in[0,1],\theta\in[0,2\pi[,y\in d(a_i)\}$$

around $d(a_i)$, where $\phi_t((u\exp(i\theta),y))$ reads $(u'\exp(i\theta),y')$ for some $u'\geq u$, for $t\geq 0$ and for $u$ small enough, so that $\theta$ is preserved by the flow. When $u$ approaches 0, the flow line through $(u\exp(i\theta),y)$ approaches $d(a_i)\cup d_\theta(A_i)$ where $d_\theta(A_i)$ is the closure of a flow line in $A_i$ determined by $\theta$, for *generic* $\theta$ (that are $\theta$ such that this closure does not end at a $b_j$). In particular, $F_\phi$ contains $\pm(d(a_i)\times A_i)$, and we examine more closely what $F_\phi$ looks like near $(d(a_i)\times h^{-1}([1,+\infty[))$.

Blow up 0 in $D^2$ to obtain an annulus $D^2(0)$. Blow up $d(a_i)$ in $D^2\times d(a_i)$ to replace $d(a_i)$ by its unit normal bundle $S^1\times d(a_i)=\{(\exp(i\theta),y)\}$. Let $D^2(0)\times d(a_i)$ denote the blown-up tubular neighborhood. Fix a fiber $D^2(0)_0=\{(u,\exp(i\theta));u\in[0,1],\exp(i\theta)\in S^1\}$ of $D^2(0)\times d(a_i)$, and its natural projection onto the disk $D^2_0=\{u\exp(i\theta)\}$. Then there are continuous embeddings

$$\begin{array}{llll}E_1\colon & D^2_0\times]-\infty,1[ & \to & h^{-1}(]-\infty,1[)\\ & (u\exp(i\theta),x) & \mapsto & m=E_1(u\exp(i\theta),x)\end{array}$$

such that $m$ is on the flow line through the point $u\exp(i\theta)$ of $D^2_0$ and $h(m)=x$, and

$$\begin{array}{llll}E_2\colon & D^2(0)_0\times]1,5[ & \to & h^{-1}(]1,5[)\\ & (u,\exp(i\theta),x) & \mapsto & n=E_2(u,\exp(i\theta),x)\end{array}$$

such that $h(n)=x$, $n$ is on the flow line through the point $u\exp(i\theta)$ of $D^2(0)_0$ if $u\neq 0$, and $E_2(0,\exp(i\theta),x)\in d_\theta(A_i)$. Then $F_\phi$ intersects $h^{-1}(]-\infty,1[)\times h^{-1}(]1,5[)$ near $d(a_i)\times h^{-1}(]1,5[)$ as the image of the continuous embedding

$$\begin{array}{llll}E\colon & D^2(0)_0\times]-\infty,1[\times]1,5[ & \to & \check{M}^2\\ & (u,\exp(i\theta),x_1,x_2) & \mapsto & (E_1(u\exp(i\theta),x_1),E_2(u,\exp(i\theta),x_2))\end{array}$$

and the boundary of $F_\phi$ contains $E(\partial_b D^2(0)_0\times]-\infty,1[\times]1,5[)$ where $\partial_b D^2(0)_0=-S^1$ is the preimage of $(0\in D^2_0)$. The closure of $]-\infty,1[$ is naturally identified with $d(a_i)$ via $E_1$, so that the boundary of $F_\phi$ contains $d(a_i)\times E_2(S^1\times]1,5[)$ and it is easy to conclude that the boundary part coming from $a_i$ near $d(a_i)\times h^{-1}([1,+\infty[)$ is $(-d(a_i))\times A_i$ (with a minor 2–dimensional abuse of notation around $a_i$). We similarly find $e(a_i)\times A_i$ in $\partial F_\phi$, and the part of $\partial F_\phi$ coming from $a_i$ is $(e(a_i)-d(a_i))\times A_i$.

For $d(b_j)$, we similarly get a part of $\partial F_\phi$

$$-\bigcup_{\exp(i\theta)\in S^1}\text{flow line } d_\theta(B_j)\times d(b_j),$$

locally oriented as (flow line $d_\theta(B_j)) \times (S^1 \times d(b_j))$ where $B_j$ locally reads $(-d_\theta(B_j) \times S^1)$, and the boundary part coming from $b_j$ is $B_j \times (e(b_j) - d(b_j))$. The two boundary parts $(e(a_i) - d(a_i)) \times A_i$ and $B_j \times (e(b_j) - d(b_j))$ intersect along a two-dimensional locus, and the 3-cycle $\partial F_\phi$ is completely described in the statement. $\diamond$

**Lemma 2.4**

$$\partial F_{\mathcal{I}} = \sum_{i=1}^{g} (d(a_i) - e(a_i)) \times A_i + \sum_{j=1}^{g} B_j \times (d(b_j) - e(b_j)) + \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c)(S^2 \times \gamma(c))$$

PROOF: The interior of a figure similar to Figure 5 embeds in the closure $A_i$ of the ascending manifold of $a_i$ in $\check{M}$. The whole closure is obtained by attaching such an open disk to the $d(b_j)$ and the $e(b_j)$.

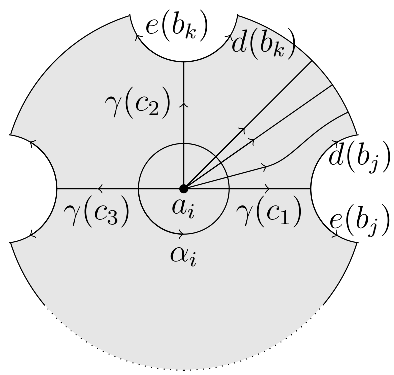


Figure 5: The interior of $A_i$ (In the figure $\sigma(c_1) = 1 = -\sigma(c_2)$.)

Recall that when the sign $\sigma(c)$ of a crossing $c \in \alpha_i \cap \beta_j$ is 1, $\beta_j$ is positively normal to $A_i$ and $\alpha_i$ is positively normal to $B_j$ along the interior of $\gamma(c)$. See Lemma 2.1.

When $A_i$ arrives at $b_j$ by a line $\gamma(c)$, it opens to $(d(b_j) - e(b_j))$ and we find

$$\partial A_i = \sum_{j=1}^{g} \sum_{c \in \alpha_i \cap \beta_j} \sigma(c)(d(b_j) - e(b_j)) = \sum_{j=1}^{g} \langle \alpha_i, \beta_j \rangle_{H_{a,2}} (d(b_j) - e(b_j))$$

$$\partial B_j = \sum_{i=1}^{g} \langle \alpha_i, \beta_j \rangle_{H_{a,2}} (d(a_i) - e(a_i)).$$

Near a connecting flow line $\gamma(c)$, $B_j$ is parametrized by $\beta_j \times \gamma(c)(]1,5[)$ and $A_i$ is parametrized by $\gamma(c)(]1,5[) \times \alpha_i$. Near the diagonal of such a line, $B_j \times A_i$ is parametrized by the height of the first point in $[1,5]$ followed by the infinitesimal difference (second point minus first point) that is parametrized by (height difference, $\alpha_i, -(-\beta_j)$), where one minus sign in front of $\beta_j$ comes

from the permutation of the parameters, and the other one comes from the fact that $\beta_j$ is now used to parametrize the difference, so that we get

$$\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(S^2\times\gamma(c))$$

in the boundary. $\diamond$

## 2.3 Using the propagator to prove Proposition 1.4

Let $\iota$ denote the continuous involution of $C_2(M)$ that exchanges two points in a pair of $(\check{M}^2 \setminus \mathrm{diag})$. Note that $\iota$ reverses the orientation of $C_2(M)$.

**Lemma 2.5** *For any* $2$*–cycle* $D=\sum_{(c,d)\in\mathcal{C}^2} f_{cd}(\gamma(c)\times\gamma(d)_{\|})$ *of* $C_2(M)$*,*

$$[D]=\left[\sum_{(c,d)\in\mathcal{C}^2} f_{cd}(\gamma(d)\times\gamma(c)_{\|})\right].$$

Proof: With the notation of Subsection 1.7, for $\varepsilon=\pm$ and $\eta=\pm$, let

$$\gamma(c)_{N^{\varepsilon}(A)N^{\eta}(B)}=\gamma^{\varepsilon}_A(c)+[c^{\varepsilon}_A,c^{\eta}_B]+\gamma^{\eta}_B(c)$$

so that

$$\gamma(c)_{\|}=\frac{1}{4}\left(\gamma(c)_{N^+(A)N^+(B)}+\gamma(c)_{N^+(A)N^-(B)}+\gamma(c)_{N^-(A)N^+(B)}+\gamma(c)_{N^-(A)N^-(B)}\right).$$

Then for any $\varepsilon$ and for any $\eta$,

$$D^{\varepsilon,\eta}=\sum_{(c,d)\in\mathcal{C}^2} f_{cd}\gamma(c)\times\gamma(d)_{N^{\varepsilon}(A)N^{\eta}(B)}$$

is a 2–cycle homotopic to

$$D^{\varepsilon,\eta}_s=\sum_{(c,d)\in\mathcal{C}^2} f_{cd}\gamma(c)_{N^{-\varepsilon}(A)N^{-\eta}(B)}\times\gamma(d).$$

Now,

$$\iota(D^{\varepsilon,\eta}_s)=-\sum_{(c,d)\in\mathcal{C}^2} f_{cd}\gamma(d)\times\gamma(c)_{N^{-\varepsilon}(A)N^{-\eta}(B)},$$

and, since $[\iota_*(S)]=-[S]$, $\iota_*$ is the multiplication by $(-1)$ in $H_2(C_2(M);\mathbb{Q})$, and $(-\iota(D^{\varepsilon,\eta}_s))$ is homologous to $D^{\varepsilon,\eta}$. Since $D$ is the average of the $D^{\varepsilon,\eta}$, and since $\left(\sum_{(c,d)\in\mathcal{C}^2} f_{cd}\gamma(d)\times\gamma(c)_{\|}\right)$ is the average of the $(-\iota(D^{\varepsilon,\eta}_s))$, the lemma is proved. $\diamond$

In order to prove Proposition 1.4, we are now left with the proof that

$$[D]=\sum_{(c,d)\in\mathcal{C}^2} f_{cd}\ell(c,d)[S].$$

We prove this by transforming the $\gamma(c)$ into

$$\gamma(c)_{N(B)} = \frac{1}{2}\left(\gamma(c)_{N^+(B)} + \gamma(c)_{N^-(B)}\right)$$

where $\gamma(c)_{N^+(B)}$ (resp. $\gamma(c)_{N^-(B)}$) is obtained from $\gamma(c)$ by pushing it infinitesimally (that is much less than slightly) in the direction of the positive (resp. negative) normal to $B_{j(c)}$ except in the neighborhood of $a_{i(c)}$ where

- $\gamma(c)_{N(B)}$ is in $A_{i(c)}$ and it is transverse to the $B_j$,
- the starting points of the $\gamma(c)_{N^+(B)}$ and the $\gamma(c)_{N^-(B)}$ such that $i(c) = i$ near $a_i$ coincide, they are denoted by $a_{i,N(B)}$,
- this starting point $a_{i,N(B)}$ does not belong to the sheets of the $B_j$ corresponding to crossings of $\alpha_i$ and the $\beta_j$, (these sheets meet along $(d(a_i) - e(a_i))$),
- the first encountered sheet from $a_{i,N(B)}$ when turning around $(d(a_i) - e(a_i))$ like $\alpha_i$ is the sheet of $c(\alpha_i)$.

See the local infinitesimal picture of Figure 6. Recall from Lemma 2.1 that $\alpha_i$ is the positive normal of $B_j$ along flow lines through positive crossings.

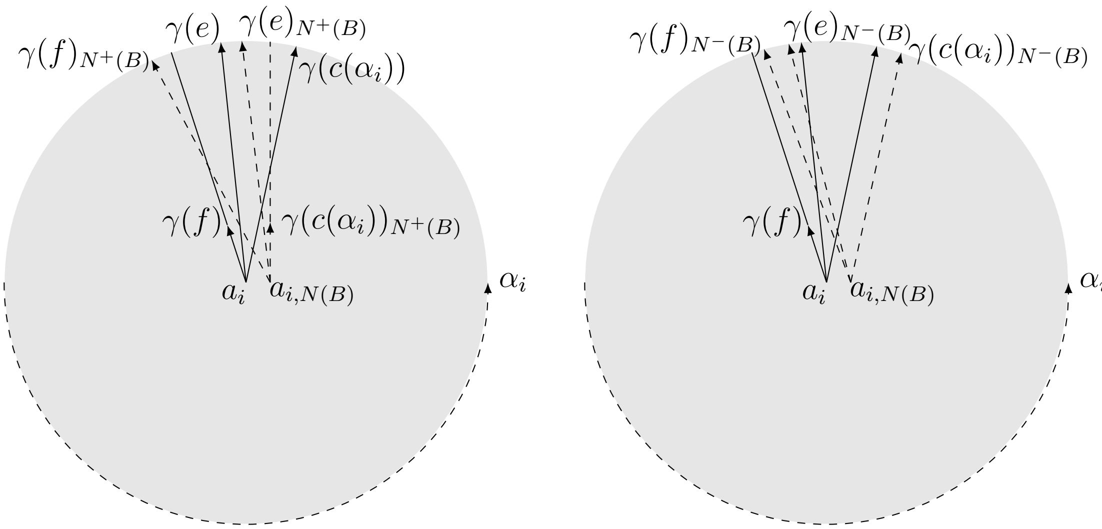


Figure 6: The $\gamma(c)_{N^+(B)}$ and the $\gamma(c)_{N^-(B)}$ near $a_i$ (where $\sigma(c(\alpha_i)) = \sigma(f) = 1 = -\sigma(e)$)

We shall similarly fix the positions of the

$$\gamma(d_{\|}) = \frac{1}{4}\left(\gamma(d)_{N^+(A)N^+(B)} + \gamma(d)_{N^+(A)N^-(B)} + \gamma(d)_{N^-(A)N^+(B)} + \gamma(d)_{N^-(A)N^-(B)}\right)$$

by homotopies of the $\gamma(d)_{N^\varepsilon(A)N^\eta(B)} = \gamma_A^\varepsilon(d) + [d_A^\varepsilon, d_B^\eta] + \gamma_B^\eta(d)$, with the notation of Subsection 1.7 so that:

- for any $d$, $\partial\gamma(d)_{N^\varepsilon(A)N^\eta(B)} = b^\eta_{j(d)} - a^\varepsilon_{i(d)}$ is fixed,
- $\gamma(d)_{N^\varepsilon(A)N^\eta(B)}$ is on the $\varepsilon$ side of $A_{i(d)}$ except near $b_{j(d)}$ where its orthogonal projection $\gamma(d)_{N^\varepsilon(A)}$ on $B_{j(d)}$ is shown in Figure 7,
- $\gamma(d)_{N^\varepsilon(A)N^\eta(B)}$ is on the $\eta$ side of $B_{j(d)}$ except near $a_{i(d)}$ where its orthogonal projection on $A_{i(d)}$ behaves like the projection of $\gamma(d)_{N^\eta(B)}$ in Figure 6 at a larger scale.

In particular, the orthogonal projections on $B_{j(d)}$ of $b^+_{j(d)}$ and $b^-_{j(d)}$ both coincide with the intersection point of the dashed segments in Figure 7, and the orthogonal projections on $A_{i(d)}$ of $a^+_{i(d)}$ and $a^-_{i(d)}$ both coincide with the intersection point of the dashed segments in Figure 6 at a larger scale.

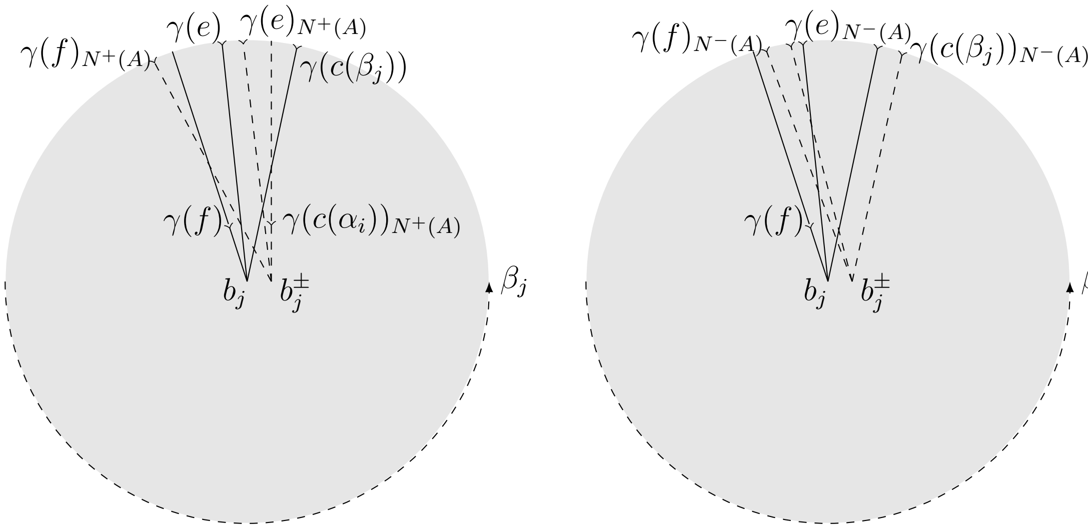


Figure 7: The orthogonal projections of the $\gamma(d)_\parallel$ on $B_j$ near $b_j$ (where $\sigma(c(\beta_j)) = \sigma(f) = 1 = -\sigma(e)$)

These positions being fixed, we have the following proposition that implies Proposition 1.4.

**Proposition 2.6**

$$\langle \gamma(c)_{N(B)} \times \gamma(d)_\parallel, F_\phi + F_{\mathcal{I}} \rangle = \ell(c,d).$$

We prove the proposition by computing the intersections with $F_{\mathcal{I}}$ and $F_\phi$.

**Lemma 2.7**

$$\langle \gamma(c)_{N(B)} \times \gamma(d)_\parallel, B_j \times A_i \rangle = -\langle [c(\alpha(c)), c|_\alpha, \beta_j \rangle \langle \alpha_i, [c(\beta(d)), d|_\beta \rangle$$

PROOF: In any case, $\langle \gamma(c)_{N(B)} \times \gamma(d)_{\|}, B_j \times A_i \rangle_{C_2(M)} = \langle \gamma(c)_{N(B)}, B_j \rangle_M \langle \gamma(d)_{\|}, A_i \rangle_M$.

The only intersection points of $\gamma(c)_{N(B)}$ with $B_j$ are shown in Figure 6. Then since the $\gamma(c)_{N(B)}$ cross the $B_j$ like the $\alpha_i$ that are positive normals for $B_j$ along flow lines associated to positive crossings

$$\langle \gamma(c)_{N(B)}, B_j \rangle_M = \langle [c(\alpha(c)), c|_{\alpha(c)}, \beta_j \rangle.$$

The computation of $\langle \gamma(d)_{\|}, A_i \rangle_M$ is similar since the position of the $\gamma(d)_{\|}$ with respect to $B_j$ does not matter. The only difference comes from the fact that the flow lines are oriented towards $b_{j(d)}$ so that they cross the $A_i$ like $(-\beta_j)$ that is the positive normal along flow lines associated to negative crossings. See Figure 7.

$$\langle \gamma(d)_{\|}, A_i \rangle_M = -\langle \alpha_i, [c(\beta(d)), d|_{\beta(d)} \rangle.$$

$\diamond$

**Lemma 2.8**

$$\langle \gamma(c)_{N(B)} \times \gamma(d)_{\|}, F_\phi \rangle = \langle [c(\alpha(c)), c|_\alpha, [c(\beta(d)), d|_\beta \rangle.$$

PROOF: Assume $c \in \alpha_i \cap \beta_{j(c)}$ and $d \in \alpha_{i(d)} \cap \beta_j$. When the first $\check{M}$-coordinate of a point of $F_\phi$ is in $\gamma(c) \setminus a_i$, its second $\check{M}$-coordinate is in $\big(\gamma(c) \cup d(b_{j(c)}) \cup e(b_{j(c)})\big)$, and therefore it is not in $\gamma(d)_{\|}$. Since the first $\check{M}$-coordinate of a point in $\gamma(c)_{N(B)} \times \gamma(d)_{\|}$ is very close to $\gamma(c)$, $\gamma(c)_{N(B)} \times \gamma(d)_{\|}$ intersects $F_\phi$ in a small neighborhood of $a_i \times A_i$.

Thus, the intersection points will be infinitely close to pairs of points on flow rays from $a_i$ on $A_i$, the closest point to $a_i$ being on $\gamma(c)_{N(B)}$ and the second one on $\gamma(d)_{\|Y'}$. Then, for a given $\gamma(c)$, the second point must be on the subsurface $D(\gamma(c))$ of $A_i$ made of the points $x$ such that the flow ray from $a_i$ to $x$ intersects $\gamma(c)_{N^+(B)}$ or $\gamma(c)_{N^-(B)}$. This interaction locus of $\gamma(c)_{N^+(B)}$, $D(\gamma(c))$, is shown in Figure 8. Since $\gamma(c)_{N^-(B)}$ is very close to $\gamma(c)_{N^+(B)}$, we can assume that $D(\gamma(c))$ is also the interaction locus of $\gamma(c)_{N^-(B)}$.

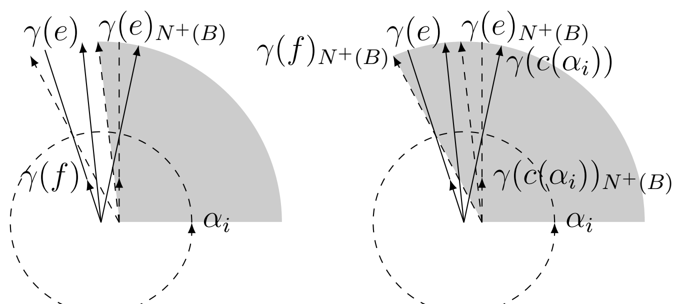


Figure 8: Interaction loci of $\gamma(e)_{N^+(B)}$ and $\gamma(f)_{N^+(B)}$ on $A_i$ (where $\sigma(f) = 1 = -\sigma(e)$)

The only intersection points of $\gamma(d)_{\|}$ with the domain $D(\gamma(c))$ of $A_i$ are near the $b_j$ and they are shown in Figure 7.

The curve $\gamma(d)_{\|}$ meets $A_i$ near a crossing line $\gamma(e)$, where *near* means in the sheet of $\gamma(e)$ around $d(b_j) \cup (-e(b_j))$,

- with probability 1 if $i(e)=i$ and if $e \in [c(\beta_j), d[_{\beta_j}$,
- with probability $1/2$ (depending on the side of $A_i$ for $\gamma(d)_{\|}$ near $b_j$) if $i(e)=i$ and if $e=d$, (this is also valid when $e=c(\beta_j)=d$),
- with probability 0 in the other cases.

The corresponding intersection point is in $D(\gamma(c))$ if $e \in [c(\alpha_i), c[_{\alpha_i}$, or if $e=c$ and $\gamma(d)_{\|}$ is on the correct side of $B_j$ (the $((-\alpha_i)$ side), that is with a probability $1/2$ independent of the previous one.

Then $M$ is oriented as (flow line $\times\ \gamma(c)_{N(B)} \times N^+(A_i)$) near $a_i$ and $F_\phi$ is oriented as (beginning of flow line $\times$ diag$(\gamma(c)_{N(B)} \times N^+(A_i)) \times$ end of flow line) that is intersected negatively by $\gamma(c)_{N(B)} \times N^+(A_i)$, where $N^+(A_i)$ is oriented like $\sigma(e)\beta_j$ and like $(-\sigma(e))\gamma(d)_{\|}$ near a point in $\left(\gamma(c)_{N(B)} \times \gamma(d)_{\|Y'}\right) \cap F_\phi$ corresponding to a crossing $e$ of $[c(\alpha(c)), c|_\alpha \cap [c(\beta(d)), d|_\beta$.

$\diamond$

# 3 Combing associated with $\mathcal{P}$

## 3.1 The combing $X(W,\mathcal{P})$ of $\check{M}$

Consider the collection $\mathcal{P}$ of favourite crossings introduced in Subsection 1.10. Up to renumbering and reorienting the $B_j$, assume that $c_i \in \alpha_i \cap \beta_i$ and that $\sigma(c_i)=1$.

There is a combing $X=X(W,\mathcal{P})$ (section of the unit tangent bundle) of $\check{M}$ that coincides with the direction $s_\phi$ of the flow (and the gradient of $h$) outside the union of regular neighborhoods $N(\gamma_i=\gamma(c_i))$ of the $\gamma_i$, that is opposite to $s_\phi$ along the interiors of the $\gamma_i$ and that is obtained as follows on $N(\gamma_i)$. Choose a natural trivialization $(X_1,X_2,X_3)$ of $T\check{M}$ on a regular neighborhood $N(\gamma_i)$ of $\gamma_i$, such that:

- $\gamma_i$ is directed by $X_1$,
- the other flow lines never have $X_1$ as an oriented tangent vector,
- $(X_1,X_2)$ is tangent to $A_i$ (except on the parts of $A_i$ near $b_i$ that come from other crossings of $\alpha_i \cap \beta_i$), and $(X_1,X_3)$ is tangent to $B_i$ (except on the parts of $B_i$ near $a_i$ that come from other crossings of $\alpha_i \cap \beta_i$).

This parallelization identifies the unit tangent bundle $UN(\gamma_i)$ of $N(\gamma_i)$ with $S^2 \times N(\gamma_i)$.

There is a homotopy $H\colon [0,1] \times (N(\gamma_i) \setminus \gamma_i) \to S^2$, such that

- $H_0$ is the unit tangent vector to the flow lines of $\phi$,
- $H_1$ is the constant map to $(-X_1)$ and

- $H_t(y)$ goes from $H_0(y) = s_\phi(y)$ to $(-X_1)$ along the shortest geodesic arc $[s_\phi(y), -X_1]$ of $S^2$ from $s_\phi(y)$ to $(-X_1)$.

Let $2\eta$ be the distance between $\gamma_i$ and $\partial N(\gamma_i)$ and let $X(y) = H(\max(0, 1 - d(y, \gamma_i)/\eta), y)$ on $N(\gamma_i) \setminus \gamma_i$, and $X = -X_1$ along $\gamma_i$.

Note that $X$ is tangent to $A_i$ on $N(\gamma_i)$ (except on the parts of $A_i$ near $b_i$ that come from other crossings of $\alpha_i \cap \beta_i$), and that $X$ is tangent to $B_i$ on $N(\gamma_i)$ (except on the parts of $B_i$ near $a_i$ that come from other crossings of $\alpha_i \cap \beta_i$). More generally, project the normal bundle of $\gamma_i$ to $\mathbb{R}^2$ in the $X_1$–direction by sending $\gamma_i$ to 0, $A_i$ to an axis $d_i(A)$ and $B_i$ to an axis $d_i(B)$. Then the projection of $X$ goes towards 0 along $d_i(B)$ and starts from 0 along $d_i(A)$, it has the direction of $s_a(y)$ at a point $y$ of $\mathbb{R}^2$ near 0, where $s_a$ is the planar reflexion that fixes $d_i(A)$ and reverses $d_i(B)$. See Figure 9.

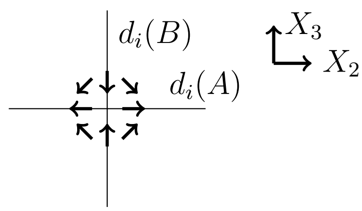


Figure 9: Projection of $X$

Then $X(y)$ is on the half great circle that contains $s_a(y)$ and $X_1$. In Figure 10 (and in Figure 3), $\gamma_i$ is a vertical segment, all the other flow lines corresponding to crossings involving $\alpha_i$ go upward from $a_i$, and $X$ is simply the upward vertical field. See also Figure 14.

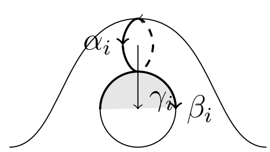


Figure 10: $\gamma_i$

## 3.2 The propagator associated with a combed Heegaard splitting

Recall that $UN(\gamma_i)$ is identified with $S^2 \times N(\gamma_i)$. Let $F_H = F_H(\mathcal{P})$ be the closure in $\partial C_2(M)$ of

$$\{(H(t,y), y) \in S^2 \times (N(\gamma_i) \setminus \gamma_i); t \in [0, \max(0, 1 - d(y, \gamma_i)/\eta)], y \in N(\gamma_i)\}.$$

**Lemma 3.1** $\partial F_H = \overline{X(\check{M})} - \overline{s_\phi(\check{M})} - \sum_{i=1}^{g} U\check{M}_{|\gamma_i}$

PROOF: We explain the $(U\check{M}_{|\gamma_i} = S^2 \times \gamma_i)$ part of $\partial F_H$, with its sign. The homotopy $H$ naturally extends to $[0,1] \times N(\gamma_i)(\gamma_i)$, where $N(\gamma_i)(\gamma_i)$ is obtained from $N(\gamma_i)$ by blowing up $\gamma_i$, so that $(-N(\gamma_i)(\gamma_i))$ contains the unit normal bundle $S^1 \times \gamma_i$ of $\gamma_i$ in $C_M$, in its boundary. Then $\partial F_H$ contains $\{(H(t,y), p_{\gamma_i}(y)) \in S^2 \times \gamma_i; t \in [0,1], y \in S^1 \times \gamma_i\}$, where $S^1$, that is the blown-up center of the fiber $D^2$ of $N(\gamma_i)$, is mapped by $s_a$ to the equator of $S^2$ so that the image of $([0,1] \times S^1)$ covers a fiber $S^2$ of $U\check{M}_{|\gamma_i}$ with degree $(-1)$. $\diamond$

Recall the 1–cycle

$$L(\mathcal{P}) = \sum_{i=1}^{g} \gamma_i - \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c)\gamma(c).$$

Let $\Sigma(\mathcal{P})$ be a two–chain bounded by $L(\mathcal{P})$ in $\check{M}$. Then let

$$F_\Sigma = U\check{M}_{|\Sigma(\mathcal{P})},$$

$F_\Sigma$ is homeomorphic to $S^2 \times \Sigma(\mathcal{P})$.

**Proposition 3.2**

$$F = F(W, \mathcal{P}) = F_\phi + F_{\mathcal{I}} + F_H + F_\Sigma$$

*is a propagator associated with the combing $X(W,\mathcal{P})$.*

PROOF: The boundary of $F$ is $(\overline{X(W,\mathcal{P})(\check{M})} + \partial_{od})$. $\diamond$

Recall that $\iota$ denotes the involution of $C_2(M)$ that exchanges two points in a pair. Then $\iota(F)$ is also a propagator associated with the combing $(-X(W,\mathcal{P}))$.

# 4 Computation of $[F_{X(W,\mathcal{P})} \cap F_{-X(W,\mathcal{P})}]$

## 4.1 A description of $[F_{X(W,\mathcal{P})} \cap F_{-X(W,\mathcal{P})}]$

Fix $W$, $\mathcal{P}$, $X = X(W,\mathcal{P})$, $L = L(\mathcal{P})$ and $\Sigma = \Sigma(\mathcal{P})$ such that $\partial\Sigma = L$.

Consider a vector field $Y$ of $X^\perp$ on $\check{M}$ such that

- $Y$ vanishes outside $C_M$,
- the norm of $Y$ is one on the $\gamma(c)$,
- $Y$ is tangent to the line $d(a_i) \cup (-e(a_i))$ at $a_i$, for any $i$ (but $Y$ does not necessarily direct the line),
- $Y$ is tangent to the line $d(b_j) \cup (-e(b_j))$ at $b_j$, for any $j$, (again, $Y$ does not necessarily direct the line),

Then $L_{\|Y}$ denotes the link parallel to $L$ obtained by pushing $L$ in the $Y$ direction. Along $\gamma(c)$, $s_a$ is the symmetry of $X^\perp$ with respect to $A_{i(c)}$ that preserves the vectors tangent to $A_{i(c)}$ and reverses the vectors tangent to $B_{j(c)}$. Similarly, define $\gamma(c) \times \gamma(d)_{\|s_a(-Y)}$ as the product of $\gamma(c)$ and a parallel of $\gamma(d)$ infinitely close to $\gamma(d)$ in the direction of $s_a(-Y)$. This can be formalised as follows. When $c \neq d$, $\gamma(c) \times \gamma(d)_{\|s_a(-Y)} = \gamma(c) \times \gamma(d)$ (away from the possibly coinciding ends). Let $[-T\gamma(c)(x), T\gamma(c)(x)]_{s_a(-Y)}$ represent a half great circle in a fiber of the unit tangent bundle of $U\check{M}_{|\gamma(c)(x)}$ through $s_a(-Y(x))$ towards the unit tangent vector $T\gamma(c)(x)$ of $\gamma(c)$, and let $s_{[-T\gamma(c),T\gamma(c)]_{s_a(-Y)}}(\gamma(c))$ be the bundle over $\gamma(c)$ of these half-circles. Then

$$\gamma(c) \times \gamma(c)_{\|s_a(-Y)} = \overline{\gamma(c)^2 \setminus \mathrm{diag}(\gamma(c)^2)} - s_{[-T\gamma(c),T\gamma(c)]_{s_a(-Y)}}(\gamma(c)).$$

In this section, we prove the following proposition.

**Proposition 4.1** *Let $Y$ be a vector field of $X^\perp$ as above. There exists a two-chain $O(a_i, b_j, s_a(-Y))$ in the hemipheres of $s_a(-Y)$ in $U\check{M}_{|\cup_i a_i \cup (\cup_j b_j)}$ such that*

$$\begin{aligned} C^i_{\uparrow\downarrow}(Y) = \; & \textstyle\sum_{(i,j,k,\ell)\in\{1,\dots,g\}^4} \mathcal{J}_{ji}\mathcal{J}_{\ell k}(B_j \cap A_k) \times (B_\ell \cap A_i)_{\|s_a(-Y)} \\ & - \textstyle\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\left(\gamma(c) \times \gamma(c)_{\|s_a(-Y)}\right) \\ & + O(a_i, b_j, s_a(-Y)) \end{aligned}$$

*is a 2–cycle of $C_2(M)$ whose homology class is unambiguously defined. Let*

$$C^b_{\uparrow\downarrow}(X,Y) = lk(L, L_{\|Y})S - \Big(X(\Sigma) - (-X)(\Sigma) - s_{[-X,X]_{s_a(-Y)}}(\partial\Sigma)\Big).$$

*Then the cycle*

$$C_{\uparrow\downarrow} = C^i_{\uparrow\downarrow}(Y) + C^b_{\uparrow\downarrow}(X,Y)$$

*represents the homology class of $F_X \cap F_{-X}$.*

## 4.2 Introduction to specific chains $F_X$ and $F_{-X}$

Let $[-1,0] \times \partial C_2(M)$ be a (topological) collar of $\partial C_2(M)$ in $C_2(M)$. Then $C_2(M)$ is homeomorphic to $\tilde{C}_2(M) = C_2(M) \setminus (]-1/2, 0] \times \partial C_2(M))$ by the *shrinking homeomorphism*

$$\begin{array}{llll} h_s\colon & C_2(M) & \to & \tilde{C}_2(M) \\ & (t,x) \in [-1,0] \times \partial C_2(M) & \mapsto & ((t-1)/2, x) \in [-1,-1/2] \times \partial C_2(M) \end{array}$$

that is the identity map outside the collar. Identifying $[-1/2, 0]$ with $[0,6]$ by the appropriate affine monotonous transformation identifies $C_2(M)$ with

$$\tilde{C}_2(M) \cup_{\partial\tilde{C}_2(M)} ([0,6] \times \partial C_2(M))$$

that is our space $C_2(M)$ from now on.

Use $h_s$ to shrink $F_\phi + F_{\mathcal{I}}$ and $\iota(F_\phi + F_{\mathcal{I}})$ into $\tilde{C}_2(M)$, and construct transverse $F_X$ and $F_{-X}$ with respective boundaries $\{6\} \times \partial F_X$ and $\{6\} \times \partial F_{-X}$ as follows:

$$\begin{aligned} F_{-X} = \ & h_s(\iota(F_\phi + F_{\mathcal{I}})) + [0,1] \times \partial\iota(F_\phi + F_{\mathcal{I}}) \\ & + \{1\} \times \iota(F_H) + [1,3] \times (\iota(-S^2 \times L + \partial_{od}) + \overline{(-X)(\check{M})}) \\ & + \{3\} \times \iota(S^2 \times \Sigma) + [3,6] \times (\overline{(-X)(\check{M})} + \iota(\partial_{od})) \end{aligned}$$

while the expression of $F_X$ will require a perturbating diffeomorphism $\Psi$ of $C_2(M)$ isotopic and very close to the identity map in order to get transversality near the diagonal,

$$\begin{aligned} F_X = \ & h_s(\Psi(F_\phi + F_{\mathcal{I}})) + [0,2] \times \partial\Psi(F_\phi + F_{\mathcal{I}}) \\ & + \{2\} \times \Psi(F_H) + [2,4] \times \Psi(-S^2 \times L + \overline{X(\check{M})} + \partial_{od}) \\ & + \{4\} \times \Psi(S^2 \times \Sigma) + [4,5] \times \Psi(\overline{X(\check{M})} + \partial_{od}) + \{5\} \times \Psi_{[\varepsilon,0]}(\partial F_X) + [5,6] \times \partial F_X \end{aligned}$$

where $\Psi_{[\varepsilon,0]}(\partial F_X)$ is the small cobordism between $\Psi(\overline{X(\check{M})} + \partial_{od})$ and $\partial F_X$ induced by the isotopy between $\Psi$ and the identity map. We describe $\Psi$ in the next subsection.

## 4.3 The perturbating diffeomorphism $\Psi_{Y,\varepsilon}$ of $C_2(M)$

Recall that $Y$ is a field like in Section 4.1. For $\eta$ small enough, we have an isotopy $\psi_Y \colon [0,\eta] \times \check{M} \to \check{M}$ such that $\frac{d}{dt}\psi_Y(t,y) = Y(y)$ and $\psi_0$ is the identity.

Let

$$\begin{array}{llll} \chi_\varepsilon\colon & [0,\varepsilon] & \to & [0,\varepsilon] \\ & 0 & \mapsto & \varepsilon \\ & \varepsilon & \mapsto & 0 \end{array}$$

be a smooth family of decreasing functions with horizontal tangents at 0 and $\varepsilon$ for $\varepsilon \in [0,\eta]$.

Fix $\varepsilon$. Consider the diffeomorphism $\Psi = \Psi_{Y,\varepsilon}$ of $C_2(\check{M})$ that is the identity outside a neighborhood $U\check{M} \times [0,\varepsilon]$ of the blown-up diagonal, where the second coordinate stands for the distance between two points in a pair and that reads

$$(v \in U\check{M}_{|m}, u) \mapsto (D\psi_Y(\chi_\varepsilon(u), m)(v), u)$$

on $U\check{M} \times [0,\varepsilon]$, so that it coincides with $D\psi$ on $(U\check{M} = U\check{M} \times \{0\})$, where $\psi = \psi_Y(\varepsilon, .)$.

Define the flow $\psi\phi\psi^{-1}$ on $\check{M}$. Observe $\Psi(\overline{s_\phi(\check{M})}) = s_{\psi\phi\psi^{-1}}(\check{M})$. The projections of the directions of the flow lines of $\psi_*(\phi) = \psi\phi\psi^{-1}$ onto a fiber of the tubular neighborhood of a line $\gamma(c)$ are shown in Figure 11. We shall refer to the directions of these projections as *horizontal* directions.

Without loss, assume that the isotopy $\psi_Y$ moves the critical points $a_i$ along $d(a_i)$ or $e(a_i)$ and the $b_j$ along the $d(b_j)$ or $e(b_j)$ (recall that $Y$ is tangent to these lines). Let $\overline{\phi}$ denote the flow $\phi$ reversed so that $\iota(F_\phi) = F_{\overline{\phi}}$.

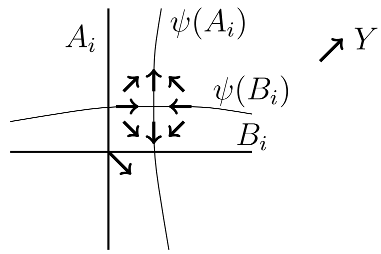


Figure 11: Horizontal directions of the flow lines of $\psi_*(\phi)$

**Lemma 4.2** *For $\varepsilon$ small enough, the direction of $\psi_*(\phi)$ along $\gamma(c)$ is very close to a geodesic arc between the direction of $\phi$ and $s_a(-Y)$, so that its distance in $S^2$ from $s_a(Y)$ is at least $\pi/4$.*

*The direction of $\overline{\phi}$ along $\psi(\gamma(c))$ is very close to a geodesic arc between the direction of $(-T(\psi(\gamma(c))))$ and $s_a(-Y)$, so that its distance in $S^2$ from $s_a(Y)$ is at least $\pi/4$.*

*Furthermore, the direction of $\psi_*(\phi)$ at the critical points and the direction of $\overline{\phi}$ at their images under $\psi$ coincide with $s_a(-Y)$.*

PROOF: The direction of $\psi_*(\phi)$ along $\gamma(c)$ is very close to the tangent direction of $\gamma(c)$ away from the ends of $\gamma(c)$ and it is slightly deviated in the orthogonal direction of $s_a(-Y)$ since $\gamma(c)$ is obtained from $\psi(\gamma(c))$ by a translation of $-Y$. See Figure 11 and Subsection 3.1. Near the critical points, the direction of $\psi_*(\phi)$ approaches the direction of $s_a(-Y)$, and it reaches it at the critical points. Similarly, the direction of $\overline{\phi}$ along $\psi(\gamma(c))$ is very close to the direction of $(-T(\gamma(c)))$ away from the ends and it is slightly deviated in the orthogonal direction of $(-s_a(Y))$. Near the critical points, the direction of $\overline{\phi}$ approaches the direction of $s_a(-Y)$, and it reaches it at the critical points. $\diamond$

**Lemma 4.3** $\lim_{\varepsilon\to 0}\Psi(F_\phi)\cap\iota(F_\phi)$ *is discrete located at the points $s_{s_a(-Y)}(a_i)$ and the $s_{s_a(-Y)}(b_j)$.*

PROOF: Observe that $F_\phi \cap \iota(F_\phi)$ is supported on the restrictions of $U\check{M}$ to the critical points. Therefore, for $\varepsilon$ small enough, $\Psi(F_\phi) \cap \iota(F_\phi)$ will be near the restrictions of $U\check{M}$ to the critical points. There are 4g points of type $s_{\overline{\phi}}(\psi(a_i))$, $s_{\psi_*(\phi)}(a_i)$, $s_{\overline{\phi}}(\psi(b_j))$ and $s_{\psi_*(\phi)}(b_j)$ in the intersection that have the wanted direction thanks to Lemma 4.2. Except for those points we have to look for flow lines for $\phi$ and flow lines for $\psi_*(\phi)$ that intersect twice and that connect the intersection points with opposite directions. Under our assumptions, this can only happen on the lines $(-e(c)\cup d(c))$ between $c$ and $\psi(c)$ for a critical point $c$. Indeed, outside $(-e(c)\cup d(c))$, $\phi$ and $\psi_*(\phi)$ both escape from the neighborhoods of $(-e(c)\cup d(c))$ if $c=a_i$, or both get closer if $c=b_i$. On these lines, the only parts where $\phi$ and $\psi_*(\phi)$ have opposite direction is between $c$ and $\psi(c)$, and the tangent direction to $\overline{\phi}$ is the direction of $s_a(-Y)$. $\diamond$

## 4.4 Reduction of the proof of Proposition 4.1

Consider a regular neighborhood $N$ of the union of the $\gamma(c)$ that contains the $\psi(\gamma(c))$, and consider the fiber bundle over $N$ whose fibers are the complement of an open disk of radius $\pi/4$ around $s_a(Y)$ in the fibers of $UN$. Let $E$ be the total space of this bundle and let $\mathcal{N} = [-1,0] \times E \subset [-1,0] \times \partial C_2(M) \subset C_2(M)$. Then $H_2(\mathcal{N};\mathbb{Z}) = 0$.

Without loss, the chains $F_X$ and $F_{-X}$ are now assumed to be transverse so that their intersection $I$ is a 2–cycle of $C_2(M)$ that we are going to compute piecewise. We shall neglect the pieces in $\mathcal{N}$ and write them as $O(\mathcal{N})$ in the statements. Sometimes, we shall also add arbitrary pieces in $\mathcal{N}$ in order to close some 2–chains and find some 2–cycle $I'$ such that

$$I' = I + O(\mathcal{N})$$

so that $I'$ will be homologous to $I$.

We shall also consider continuous limits when possible to simplify the expressions like in Lemma 4.3 that now reads:

$$\lim_{\varepsilon \to 0} \Psi(F_\phi) \cap \iota(F_\phi) = O(\mathcal{N})$$

or,
for $\varepsilon > 0$ small enough, $\Psi(F_\phi) \cap \iota(F_\phi) = O(\mathcal{N})$.

For example,

$$\begin{aligned} F_X \cap F_{-X} \cap ([5/2,6] \times \partial C_2(M)) &= [5/2,3] \times (\psi_*(X)(L) - (-X)(\psi(L))) \\ &\quad +\{3\} \times (-\psi_*(X)(\Sigma) + S^2 \times (\psi(L) \cap \Sigma)) \\ &\quad -[3,4] \times (-X)(\psi(L)) \\ &\quad +\{4\} \times (-X)(\psi(\Sigma)) \\ &= \{3\} \times (-\psi_*(X))(\Sigma) + \{4\} \times (-X)(\psi(\Sigma)) \\ &\quad +S^2 \times (\psi(L) \cap \Sigma) + O(\mathcal{N}) \end{aligned}$$

Then $S^2 \times (\psi(L) \cap \Sigma)$ is a disjoint union of spheres homologous to $lk(L, L_{\parallel Y})[S]$. Let

$$\ell = \lim_{\varepsilon \to 0} (-\{3\} \times (\psi_*(X))(\Sigma) + \{4\} \times (-X)(\psi(\Sigma)))$$

$$\begin{aligned} \ell &= -\{3\} \times X(\Sigma) + \{4\} \times (-X)(\Sigma) \\ &= -\{3\} \times X(\Sigma) + \{4\} \times (-X)(\Sigma) - [3,4] \times (-X)(L) + \{3\} \times s_{[-X,X]_{s_a(-Y)}}(L) + O(\mathcal{N}) \end{aligned}$$

Then $F_X \cap F_{-X} \cap ([5/2,6] \times \partial C_2(M))$ is homologous to $C^b_{\uparrow\downarrow}(X,Y)$ mod $\mathcal{N}$ and the proof of Proposition 4.1 is reduced to the proof of the two following propositions.

**Proposition 4.4**

$$F_X \cap F_{-X} \cap \tilde{C}_2(M) = C^i_{\uparrow\downarrow}(Y) + O(\mathcal{N}).$$

**Proposition 4.5**

$$F_X \cap F_{-X} \cap ([0,5/2] \times \partial C_2(M)) = O(\mathcal{N}).$$

## 4.5 Proof of Proposition 4.4

**Lemma 4.6**

$$\lim_{\varepsilon \to 0} \Psi(F_{\mathcal{I}}) \cap \iota(F_{\mathcal{I}}) = \sum_{(i,j,k,\ell) \in \{1,\dots,g\}^4} \mathcal{J}_{ji}\mathcal{J}_{\ell k}(B_j \cap A_k) \times (B_\ell \cap A_i)_{\| s_a(-Y)} + O(\mathcal{N}).$$

Proof: The intersection $B_j \times A_i \cap (A_k \times B_\ell)$ is cooriented by the positive normals of $B_j$, $A_i$, $A_k$ and $B_\ell$ in this order. Therefore the intersection reads like in the statement away from the diagonal. Near the diagonal and away from the critical points, $A_i$ and $B_j$ are moved in the direction of $Y$. If $Y = \vec{a} + \vec{b}$ where $\vec{a}$ is tangent to $A_k$ and $\vec{b}$ is tangent to $B_\ell$, then abusively write $A_i = A_k + \vec{b}$ and $B_j = B_\ell + \vec{a}$ and see that the difference of the two points is moved in the direction $(\vec{b} - \vec{a})$ of $s_a(-Y)$, so that the corresponding intersection sits inside the neglected part $\mathcal{N}$. (When two points vary along the same $\gamma(c)$, the second one will be deviated in the direction of $s_a(-Y)$ so that the limit pair of points describe an arc in $U\check{M}_{|\gamma(c)}$ from $-T\gamma(c)$ to $T\gamma(c)$ through $s_a(-Y)$, that is along the half great circle $[-T\gamma(c), T\gamma(c)]_{s_a(-Y)}$.)

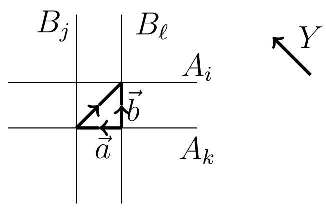


Figure 12: Deviation near the diagonal

Near a critical point $P$, two points can come from different crossings. Then the direction between them in $\overline{(B_j \cap A_k) \times (B_\ell \cap A_i) \setminus \text{diag}}$ is orthogonal to $Y = \pm s_a(Y)$. The field $Y$ can be assumed to preserve the $B$-sheets near the $a_i$ and the $A$-sheets near the $b_j$. Then the difference of the two points is moved in the direction of $s_a(-Y)$ so that it belongs to the hemisphere of $s_a(-Y)$. $\diamond$

**Lemma 4.7**

$$\lim_{\varepsilon \to 0} \Psi(F_\phi) \cap \iota(F_{\mathcal{I}}) = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)}(-\sigma(c))\overline{\{(\gamma(c)(t_1), \gamma(c)(t_2)); t_1 < t_2\}} + O(\mathcal{N}).$$

Proof: The intersection $F_\phi \cap \left(\iota(F_{\mathcal{I}}) = \sum_{(i,j) \in \{1,\dots,g\}^2} \mathcal{J}_{ji} A_i \times B_j\right)$ is supported on the

$$\overline{\{(\gamma(c)(t_1), \gamma(c)(t_2)); t_1 < t_2\}}$$

away from the unit bundles of the critical points. It is transverse except near these unit bundles.

Let $c \in \alpha_i \cap \beta_j$. Along $\gamma(c)$, $A_i \times B_j$ is cooriented by $\beta_j \times \alpha_i$. Then $F_\phi \cap (A_i \times B_j)$ will be oriented as $(-\sigma(c))\{(\gamma(c)(t_1), \gamma(c)(t_2)); t_1 < t_2\}$. Since $\psi_*(\phi)$ is almost vertical away from the critical points, we are left with the behaviour near the critical points. Near $a_i$ on $A_i$, (or near $b_j$ on $B_j$) the direction of $\psi_*(\phi)$ is in the hemisphere of $s_a(-Y)$, according to Lemma 4.2, so that the pairs of points of $A_i \times B_j$ connected by flow lines of $\psi_*(\phi)$ near a critical point are in $\mathcal{N}$. $\diamond$

Similarly, we have

**Lemma 4.8**

$$\lim_{\varepsilon \to 0} \Psi(F_{\mathcal{I}}) \cap \iota(F_\phi) = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)}(-\sigma(c))\overline{\{(\gamma(c)(t_1), \gamma(c)(t_2)); t_1 > t_2\}} + O(\mathcal{N}).$$

Proof: Away from the unit bundles of the critical points, it is clear. According to Lemma 4.2, the direction of $\overline{\phi}$ on $\psi(A_i)$ near $\psi(a_i)$ (or on $\psi(B_j)$ near $\psi(b_j)$) is in the hemisphere of $s_a(-Y)$, so that the pairs of points of $(\psi(B_j) \times \psi(A_i)) \cap \iota(F_\phi)$ near the critical points are again in $\mathcal{N}$. $\diamond$

Proposition 4.4 is a direct corollary of Lemmas 4.3, 4.6, 4.7, 4.8. $\diamond$

## 4.6 Proof of Proposition 4.5

We prove that $(F_X \cap F_{-X} \cap ([0, 5/2] \times \partial C_2(M)))$ is in $\mathcal{N}$.

According to Theorem 2.2,

$$\partial(F_\phi + F_{\mathcal{I}}) = \partial_{od} + \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(S^2 \times \gamma(c)) + \overline{s_\phi(\check{M})}.$$

Therefore, according to Lemmas 4.3 and 4.2,

$$\Psi(\partial(F_\phi + F_{\mathcal{I}})) \cap \partial\iota(F_\phi + F_{\mathcal{I}}) = O(\mathcal{N}).$$

Let us now show that

$$\Psi(\partial(F_\phi + F_{\mathcal{I}})) \cap \iota(F_H) = O(\mathcal{N}).$$

According to the construction of $F_H$ in Subsections 3.1 and 3.2, $\iota(F_H)$ intersects $\Psi(S^2 \times \gamma(c)) = S^2 \times \psi(\gamma(c))$ on $s_{[\overline{\phi},-X]}(\psi(\gamma(c))$ where $[\overline{\phi}, -X]$ is the shortest geodesic arc between the tangent to $\overline{\phi}$ and $-X$, that is in the hemisphere of $s_a(-Y)$, according to Lemma 4.2. Now, look at the intersection of $\iota(F_H)$ and $s_{\psi_*(\phi)}(\check{M})$, where the direction of $\psi_*(\phi)$ must belong to $[\overline{\phi}, -X]$. This can only happen in a tubular neighborhood of $\gamma_i$ at a place where the flow lines of $\psi_*(\phi)$ and $\overline{\phi}$ have the same horizontal direction. This only happens between $\gamma_i$ and $\psi(\gamma_i)$, more precisely in the preimage of the rectangle shown in Figure 13 under the orthogonal projection directed by $X_1$. There the horizontal direction is close to the direction of $s_a(-Y)$.

Similarly,

$$\Psi(F_H) \cap \left(S^2 \times L + \overline{(-X)(\check{M})}\right) = O(\mathcal{N}).$$

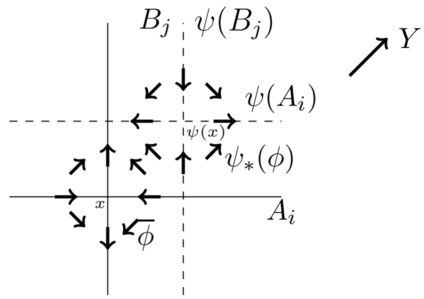


Figure 13: Tangencies of the flow lines of $\overline{\phi}$ and $\psi_*(\phi)$ near a $\gamma(c)$

Indeed, since the horizontal component of the direction of $\psi_*(\phi)$ along $\gamma(c)$ is in the direction of $s_a(-Y)$, $\Psi(F_H) \cap (S^2 \times L) = O(\mathcal{N})$. Now, $(-X)$ can belong to $[\psi_*(\phi), \psi_*(X)]$ if the horizontal component of $(-X)$ that is the horizontal component of the tangent to $\overline{\phi}$ and the horizontal component of $\psi_*(\phi)$ have the same direction. This can only happen in the same rectangles as before where $(-X)$ is in the hemisphere of $s_a(-Y)$. $\diamond$

# 5 Concluding the proof of Theorem 1.5

Recall that $W$, $\mathcal{P}$, $X = X(W,\mathcal{P})$, $L = L(\mathcal{P})$ and $\Sigma$ such that $\partial\Sigma = L$ are fixed. Note that $X$ depends neither on the orientations of the $\alpha_i$ and the $\beta_j$, nor on their order. Furthermore $e(W,\mathcal{P})$ is independent of the order of the $\beta_j$. Thus, the permutation $\rho$ of $\{1,2,\dots,g\}$ associated with $\mathcal{P}$ is assumed to be the identity, without loss.

## 5.1 Reducing the proof of Theorem 1.5 to an Euler class computation

Consider the four following fields $Y^{++}$, $Y^{+-}$, $(Y^{-+} = -Y^{+-})$ and $(Y^{--} = -Y^{++})$ in a neighborhood of the $\gamma(c)$. $Y^{++}$ and $Y^{+-}$ are positive normals for $A_i$ on $H_{a,\leq 3} = C_M \cap h^{-1}(]-\infty,3])$, and $Y^{++}$ and $Y^{-+}$ are positive normals for $B_j$ on $H_{b,\geq 3} = C_M \cap h^{-1}([3,+\infty[)$. Then with the notation of Subsections 1.7 and 1.10,

$$lk(L(\mathcal{P}), L(\mathcal{P})_{\parallel}) = \frac{1}{4} \sum_{(\varepsilon,\eta)\in\{+,-\}^2} lk(L, L_{\parallel Y^{\varepsilon,\eta}})$$

and, with the notation of Proposition 4.1,

$$[C_{\uparrow\downarrow}] = \frac{1}{4} \sum_{(\varepsilon,\eta)\in\{+,-\}^2} \left[C^i_{\uparrow\downarrow}(Y^{\varepsilon,\eta}) + C^b_{\uparrow\downarrow}(X, Y^{\varepsilon,\eta})\right]$$

where $s_a(-Y^{\varepsilon,\eta})=Y^{\varepsilon,(-\eta)}$, so that the collections of the $s_a(-Y^{\varepsilon,\eta})$ is the same as the collection of the $Y^{\varepsilon,\eta}$ and, thanks to Lemma 2.1,

$$[C_h]=\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}[C^i_{\uparrow\downarrow}(Y^{\varepsilon,\eta})].$$

Therefore, thanks to Proposition 4.1, the proof of Theorem 1.5 is reduced to the proof of the following equality.

$$\left[X(\Sigma)-(-X)(\Sigma)-\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}s_{[-X,X]_{Y^{\varepsilon,\eta}}}(\partial\Sigma)\right]=e(W,\mathcal{P})[S].$$

Consider the rank 2 sub-vector bundle $X^{\perp}$ of $T\check{M}$ of the planes orthogonal to $X$. Let $X^{\perp}(\Sigma)$ be the total space of the restriction of $X^{\perp}$ to our surface $\Sigma$. Let $Y$ be a non-vanishing section of $X^{\perp}$ on $\partial\Sigma$. The *relative Euler class* $e(X^{\perp}(\Sigma),Y)$ *of* $Y$ *in* $X^{\perp}(\Sigma)$ is the obstruction to extending $Y$ as a nonzero section of $X^{\perp}(\Sigma)$ over $\Sigma$. If $\tilde{Y}$ is an extension of $Y$ as a section of $X^{\perp}(\Sigma)$ transverse to the zero section $s_0(X^{\perp}(\Sigma))$, then

$$e(X^{\perp}(\Sigma),Y)=\langle\tilde{Y}(\Sigma),s_0(X^{\perp}(\Sigma))\rangle_{X^{\perp}(\Sigma)}.$$

**Lemma 5.1** *Under the assumptions above,*

$$\left[X(\Sigma)-(-X)(\Sigma)-s_{[-X,X]_Y}(\partial\Sigma)\right]=e(X^{\perp}(\Sigma),Y)[S]$$

*in* $H_2(C_2(M))$.

PROOF: If $Y$ extends as a nonzero section of $X^{\perp}(\Sigma)$ still denoted by $Y$, then the cycle of the left-hand side bounds $s_{[-X,X]_Y}(\Sigma)$. This allows us to reduce the proof to the case when $\Sigma$ is a neighborhood of a zero of the extension $\tilde{Y}$ above, that is when $\Sigma$ is a disk $\Delta$ equipped with a trivial $D^2$-bundle, and when $Y\colon\partial\Delta\to\partial D^2$ has degree $d=\pm1$. Then $d=e(X^{\perp}(\Delta),Y)$, and $\left[X(\Delta)-(-X)(\Delta)-s_{[-X,X]_Y}(\partial\Delta)\right]=d[S]$. ◇

Thus,

$$\left[X(\Sigma)-(-X)(\Sigma)-\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}s_{[-X,X]_{Y^{\varepsilon,\eta}}}(\partial\Sigma)\right]=\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X^{\perp}(\Sigma),Y^{\varepsilon,\eta})[S].$$

The proof of Theorem 1.5 is now reduced to the proof of the following proposition that occupies the rest of this section.

**Proposition 5.2**
$$e(W,\mathcal{P})=\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(W,\mathcal{P})^{\perp}(\Sigma),Y^{\varepsilon,\eta}).$$

**Remark 5.3** Note that this proposition provides a combinatorial formula for the average of the Euler classes in the right-hand side. In this formula, the $d_e(\beta_j)$ and $d_e(|c_{j(c)},c|_\beta)$ depend on our picture of the Heegaard diagram in Figure 2. Thus, the proposition implies that the sum $e(W,\mathcal{P})$ is independent of our special picture of the Heegaard diagram.

## 5.2 A surface $\Sigma(L(\mathcal{P}))$

Let $H_{b,\geq 2}=C_M\cap h^{-1}([2,+\infty[)$. For any crossing $c$ of $\mathcal{C}$, define the triangle $T_\beta(c)$ in the disk $(D_{\geq 2}(\beta_{j(c)})=B_{j(c)}\cap H_{b,\geq 2})$ such that
$$\partial T_\beta(c)=[c_{j(c)},c]_\beta+(\gamma(c)\cap H_{b,\geq 2})-(\gamma_{j(c)}\cap H_{b,\geq 2}).$$
Similarly, define the triangle $T_\alpha(c)$ in the disk $(D_{\leq 2}(\alpha_{i(c)})=A_{i(c)}\cap H_a)$ such that
$$\partial T_\alpha(c)=-[c_{i(c)},c]_\alpha+(\gamma(c)\cap H_a)-(\gamma_{i(c)}\cap H_a).$$

**Proposition 5.4** *There exists a 2-chain $F(\mathcal{P})$ in $H_{a,2}$ such that the boundary of*
$$\begin{aligned}\Sigma(L(\mathcal{P}))= \;& F(\mathcal{P})-\textstyle\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)(T_\beta(c)+T_\alpha(c))\\ &+\textstyle\sum_{(j,i)\in\{1,\dots,g\}^2}\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{ji}\left(\langle\alpha_i,|c_{j(c)},c|_\beta\rangle D_{\geq 2}(\beta_j)-\langle|c_{i(c)},c|_\alpha,\beta_j\rangle D_{\leq 2}(\alpha_i)\right)\end{aligned}$$
*is $L(\mathcal{P})$.*

PROOF: The boundary of the defined pieces reads $(L(\mathcal{P})+u)$ where the cycle $u$ is
$$\begin{aligned}u=\;&\textstyle\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\left([c_{i(c)},c]_\alpha-[c_{j(c)},c]_\beta\right)\\ &+\textstyle\sum_{(j,i)\in\{1,\dots,g\}^2}\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{ji}\left(\langle\alpha_i,|c_{j(c)},c|_\beta\rangle\beta_j-\langle|c_{i(c)},c|_\alpha,\beta_j\rangle\alpha_i\right).\end{aligned}$$
Compute $\langle\alpha_k,u\rangle$, by pushing $u$ in the direction of the positive normal to $\alpha_k$ and in the direction of the negative normal, and by averaging, so that locally
$$\langle\alpha_k,|c_{i(c)},c|_\alpha-|c_{j(c)},c|_\beta\rangle=-\langle\alpha_k,|c_{j(c)},c|_\beta\rangle$$
and
$$\langle\alpha_k,u\rangle=-\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\langle\alpha_k,|c_{j(c)},c|_\beta\rangle+\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\langle\alpha_k,|c_{j(c)},c|_\beta\rangle=0.$$
Similarly, $\langle u,\beta_\ell\rangle=0$ for any $\ell$ so that $(-u)$ bounds a 2-chain $F(\mathcal{P})$ in $H_{a,2}$. $\diamond$

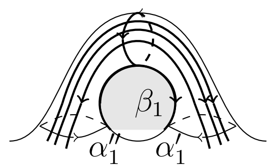

Figure 14: How the $\beta_j$ look like near the handles' cores

## 5.3 Proof of the combinatorial formula for the Euler classes

In this section, we prove Proposition 5.2.

Represent $H_a$ like in Figure 3, and assume that the curves $\beta_j$ intersect the handles as arcs parallel to Figure 14, one below through the favourite crossing and the other ones above.

Then cut this upper neighborhood of the cores of the handles in order to get the planar picture of the Heegaard diagram of Figure 2, Subsection 1.10.

Let $H^{\mathcal{P}}_{a,2}$ denote the complement of disk neighborhoods of the favourite crossings in the surface $H_{a,2}$. See $H^{\mathcal{P}}_{a,2}$ as the surface obtained from the rectangle of Figure 2 by adding a band of the handle upper part of each $\alpha_i$ so that the band of $\alpha_i$ contains all the non-favourite crossings of $\alpha_i$. See Figure 15 for an immersion of this surface in the plane.

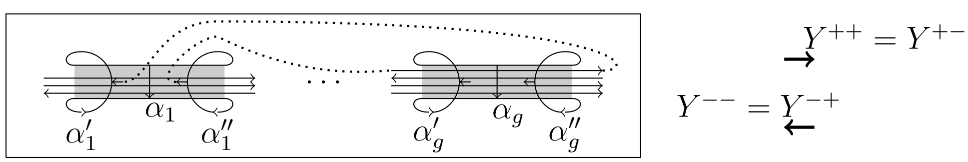

Figure 15: The punctured surface $H^{\mathcal{P}}_{a,2}$

Extend every $Y = Y^{\varepsilon,\eta}$ on $H_a$ so that the fields $Y^{\varepsilon,\eta}$ are horizontal and their projections are the depicted constant fields in Figure 15.

Note that $[0, 2g] \times [0, 4] \times [-\infty, 0]$ is the product of Figure 16 by $[-\infty, 0]$ where all the flow lines are directed by $[-\infty, 0]$.

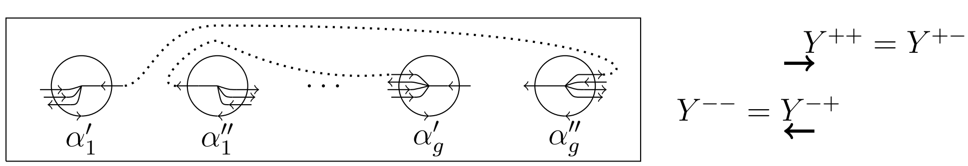

Figure 16: A typical slice of $[0, 2g] \times [0, 4] \times [-\infty, 0]$

Similarly, assume that the $\alpha$-curves are orthogonal to the picture on the lower parts of the handles in the standard picture of $H_b$ in Figure 3, and draw a planar picture similar to Figure 15

of $H^{\mathcal{P}}_{a,4}$ (that is $h^{-1}(4)\cap C_M$ minus disk neighborhoods of the favourite crossings), by starting with Figure 17 and by adding a vertical band cut by an horizontal arc of $\beta_j$ oriented from right to left, for each $\beta_j$.

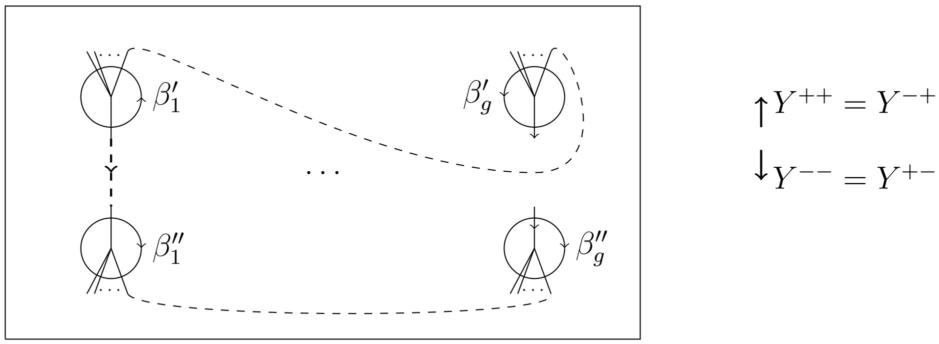


Figure 17: A typical slice of $[0,2g]\times[0,4]\times[6,\infty]$

Again, $[0,2g]\times[0,4]\times[6,\infty]$ is the product of Figure 17 by $[6,\infty]$ where all the flow lines are directed by $[6,\infty]$. Extend every $Y=Y^{\varepsilon,\eta}$ on $H_b$ so that $Y$ looks constant and horizontal in our standard figure of $H_b$ in Figure 3 and so that its projection on Figure 17 is the drawn constant field.

Also assume that every $Y=Y^{\varepsilon,\eta}$ varies in a quarter of horizontal plane in our tubular neighborhoods of the $\gamma_i$ in Figure 10. Similarly, extend every $Y=Y^{\varepsilon,\eta}$ in the product by $[2,4]$ of the bands of Figure 15 so that $Y^{\varepsilon,\eta}$ is horizontal and is never a $(-\varepsilon)$-normal of the $A_i$ there.

Let $H^{\mathcal{C}}_{a,2}$ denote the punctured rectangle of Figure 2, that is a subsurface of $H_{a,2}$. Now, $Y$ is defined everywhere except in $H^{\mathcal{C}}_{a,2}\times]2,4[$ so that

$$\begin{array}{rcl} e(X^{\perp}(\Sigma),Y) & = & e(X^{\perp}(\Sigma\cap(H^{\mathcal{C}}_{a,2}\times[2,4])),Y)\\ & = & -\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)e(X^{\perp}([c_{j(c)},c]_{\beta}\times[2,4])),Y)\\ & & +\sum_{(j,i)\in\{1,\dots,g\}^2}\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{ji}\langle\alpha_i,|c_{j(c)},c|_{\beta}\rangle e(X^{\perp}(\beta_j\times[2,4]),Y).\end{array}$$

for the surface $\Sigma$ of Proposition 5.4. Thus, Proposition 5.2 will be proved as soon as we have proved the following lemma.

**Lemma 5.5** *With the notation of Subsection 1.10,*

$$d_e(\beta_j) = -\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X^{\perp}(\beta_j\times[2,4]),Y^{\varepsilon,\eta})$$

*and*

$$d_e(|c_{j(c)},c|_{\beta}) = -\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X^{\perp}(|c_{j(c)},c|_{\beta}\times[2,4]),Y^{\varepsilon,\eta}).$$

Proof: Consider an arc $[c,d]_\beta$ between two consecutive crossings of $\beta$. Let $[c',d'] = [c,d]_\beta \cap H^{\mathcal{C}}_{a,2}$. On $[c',d']\times[2,4]$, the field $X$ is directed by $[2,4]$, the field $Y$ is defined on $\partial\left([c',d']\times[2,4]\right)$, and it is in the hemisphere of the $\eta$-normal of $[c',d']\times[2,4]$ along $\partial\left([c',d']\times[2,4]\right)\setminus[c',d']\times\{2\}$ (the $\eta$-normal is the positive normal when $\eta=+$ and the negative normal otherwise). Then $e(X^\perp([c',d']\times[2,4]),Y^{\varepsilon,\eta})$ is the degree of $Y^{\varepsilon,\eta}$ at the $(-\eta)$-normal of $[c',d']=[c',d']\times\{2\}$, in the fiber of the unit tangent bundle of $H_{a,2}$ trivialised by the normal to $[c',d']$. Thus, $e(X^\perp([c',d']\times[2,4]),Y^{\varepsilon,\eta})$ is the opposite of the degree of the $(-\eta)$-normal of the curve in the fiber of $H_{a,2}$ at $Y^{\varepsilon,\eta}$ trivialised by $Y^{\varepsilon,\eta}$ (that is by Figure 2) along $[c',d']$. This $(-\eta)$-normal starts and ends as vertical in this figure, and $Y^{\varepsilon,\eta}$ is horizontal with a direction that depends on the sign of $\varepsilon$. The $(-\eta)$-normal to $[c',d']$ makes $(d_e(|c,d|_\beta)\in\frac{1}{2}\mathbb{Z})$ positive loops with respect to the parallelization induced by Figure 2. Therefore the sum of the degree of the $(-\eta)$ normal at the direction of $Y^{\varepsilon,\eta}$ and at the direction of $Y^{(-\varepsilon),\eta}$ is $2d_e(|c,d|_\beta)$.

This shows that

$$d_e(|c,d|_\beta) = -\frac{1}{2}\left(e(X^\perp([c',d']\times[2,4]),Y^{\varepsilon,\eta}) + e(X^\perp([c',d']\times[2,4]),Y^{(-\varepsilon),\eta})\right)$$

and allows us to conclude the proof of Lemma 5.5, and therefore the proofs of Proposition 5.2 and Theorem 1.5. $\diamond$

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